D.K.T.E. SOCIETY'S

TEXTILE AND ENGINEERING INSTITUTE, ICHALKARANJI.

(An Autonomous Institute.)

FIRST YEAR, B.TECH

GET102- ENGINEERING MATHEMATICS-I

TUTORIAL NO. : 09

TOPIC : EXPANSION OF FUNCTIONS, PARTIAL DIFFERENTIATION & APPLICATIONS

TITLE : PARTIAL DIFFERENTIATION & APPLICATIONS

TUTORIAL NO.09

TOPIC: Expansion of functions, Partial Differentiation and Applications.

TITLE: Partial Differential Equations and its Applications.

1. If $u = (1-2xy+y^2)^{\frac{-1}{2}}$ then prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$.
2. If $u = \log(x^3+y^3+z^3-3xyz)$ then prove that $(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z})^2 u = -\frac{9}{(x+y+z)^2}$.
3. If $z = x^2+y^2$, $x = at\sin t$, $y = at\cos t$, verify that $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.
4. If $z = f(x,y), u = e^x$, $v = e^y$ prove that $\frac{\partial^2 z}{\partial x\partial y} = uv\frac{\partial^2 z}{\partial u\partial v}$.
5. If $z = f(x,y), x = e^u\cos v$, $y = e^u\sin v$, prove that $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$.
6. If $z = f(x,y)$, & $u = lx+my$, $v = ly-mx$ then show that $(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 = (l^2+m^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$.
7. If $x^m + y^m = a^m$ then find $\frac{d^2y}{dx^2}$.
8. If $x^xy^yz^z = c$, then show that at $x = y = z$, $\frac{\partial^2 z}{\partial x\partial y} = -[x\log ex]^{-1}$.
9. If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, then prove that $\frac{\partial u}{\partial y} = \frac{-x}{y}\frac{\partial u}{\partial x}$.
10. If $u = x^2\log[\frac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}]$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2\log[\frac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}]$.
11. If $u = x^nf\left(\frac{y}{x}\right) + y^nf\left(\frac{x}{y}\right)$ then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$.
12. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.
13. If $u = \log[\frac{x^3+y^3}{x^2+y^2}]$ then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -1$.
14. Show that JJ'=1 for $x = e^v\sec u$, $y = e^v\tan u$.
15. If $x = v^2+w^2$, $y = w^2+u^2$, $z = u^2+v^2$, prove that JJ'=1.
16. If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ then prove that JJ'=1.
17. If $x+y+z = u$, $y+z = uv$, $z = uvw$, then prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.
18. Find the minimum and maximum values of $x^4+y^4-2x^2+4xy-2y^2$.
19. Find the minimum and maximum values of $x^3+3xy^2-3x^2-3y^2+4$.
20. Examine for stationary values, $f(x,y) = \sin x + \sin y + \sin(x+y)$.

Topic Name - Expansion of f^ns, Partial diff. & it's applications.

Tut - Partial differentiation & it's Applications

(1) If $u = (1-2xy+y^2)^{-1/2}$ then prove that

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3.$$

⇒ Solution - since $u = (1-2xy+y^2)^{-1/2}$ —— (1)

diff. partially w.r.t 'x'

$$\therefore \frac{\partial u}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-1/2-1}\cdot(-2y)$$

$$\therefore \frac{\partial u}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2y)$$

$$\therefore \frac{\partial u}{\partial x} = y\cdot(1-2xy+y^2)^{-3/2}$$

$$= y\left[(1-2xy+y^2)^{-1/2}\right]^3$$

$$\boxed{\frac{\partial u}{\partial x} = y\cdot u^3} \longrightarrow \text{from (1)}$$

Also diff. eq^n (1) partially w.r.t 'y'

$$\therefore \frac{\partial u}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-1/2-1}\cdot(-2x+2y)$$

$$\therefore \frac{\partial u}{\partial y} = x(1-2xy+y^2)^{-3/2} - y(1-2xy+y^2)^{-3/2}$$

$$\frac{\partial u}{\partial y} = x\left[(1-2xy+y^2)^{-1/2}\right]^3 - y\left[(1-2xy+y^2)^{-1/2}\right]^3$$

$$\therefore \boxed{\frac{\partial u}{\partial y} = xu^3 - yu^3} \longrightarrow \text{from (1)}$$

Now $$x\left(\frac{\partial u}{\partial x}\right) - y\left(\frac{\partial u}{\partial y}\right) = x(yu^3) - y(xu^3) + y^2u^3$$

$$= xyu^3 - xyu^3 + y^2u^3$$

$$= 0 + y^2u^3$$

$$\boxed{x\left(\frac{\partial u}{\partial x}\right) - y\left(\frac{\partial u}{\partial y}\right) = y^2u^3}$$

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(2) If $u = \log(x^3+y^3+z^3-3xyz)$ then prove that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

$\Rightarrow$ solution - given $u = \log(x^3+y^3+z^3-3xyz)$ — (1)

diff. (1) partially w.r.t 'x', 'y' & 'z'

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\cdot(3x^2-3yz) \quad \text{— (2)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}\cdot(3y^2-3xz) \quad \text{— (3)}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}\cdot(3z^2-3xy) \quad \text{— (4)}$$

Now

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2-yz)+3(y^2-xz)+3(z^2-xy)}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-xy-yz-xz)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-xz)}}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{(x+y+z)} = v \quad \text{consider}$$

$$\therefore \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot v$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z} \quad \text{— (5)}$$

$$v = \frac{-3}{(x+y+z)}$$

$$\therefore \frac{\partial v}{\partial x} = \frac{-3}{(x+y+z)^2}, \quad \frac{\partial v}{\partial y} = \frac{-3}{(x+y+z)^2}, \quad \frac{\partial v}{\partial z} = \frac{-3}{(x+y+z)^2}$$

$\therefore$ eq^n 1 becomes.

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

$$= RHS$$

If $z = x^2+y^2$, $x = at\sin t$, $y = at\cos t$, verify that $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

solution – given $z = x^2+y^2$

$$\therefore \quad z = (at\sin t)^2 + (at\cos t)^2$$

$$\therefore \quad z = a^2t^2\sin^2 t + a^2t^2\cos^2 t$$

$$\therefore \quad z = a^2t^2(\sin^2 t + \cos^2 t)$$

$$\therefore \quad z = a^2t^2(1) = a^2t^2$$

diff. z w.r.t 't'

$$\therefore \frac{dz}{dt} = 2a^2t. \quad \text{—— (1)}$$

we know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

$$\therefore \frac{dz}{dt} = [2x][a\sin t + at\cos t] + [2y][a\cos t - at\sin t]$$

$$\therefore \frac{dz}{dt} = [2at\sin t][a\sin t + at\cos t] + [2at\cos t][a\cos t - at\sin t]$$

$$= 2a^2t\sin^2 t + \cancel{2a^2t^2\sin t\cdot\cos t} + 2a^2t\cos^2 t - \cancel{2a^2t^2\sin t\cos t}$$

$$= 2a^2t\sin^2 t + 2a^2t\cos^2 t$$

$$= 2a^2t[\sin^2 t + \cos^2 t]$$

$$= 2a^2(t)(1) \qquad (\text{as } \sin^2 t + \cos^2 t = 1)$$

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(4) If $z = f(x,y)$, $u = e^x$, $v = e^y$, prove that
$$\frac{\partial^2 z}{\partial x \cdot \partial y} = u \cdot v \frac{\partial^2 z}{\partial u \cdot \partial v}$$

$\Rightarrow$ solution -

we know that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot e^x + \frac{\partial z}{\partial v} \cdot 0^y$$

$$\therefore \frac{\partial z}{\partial x} = e^x \frac{\partial z}{\partial u}.$$

$$\therefore \frac{\partial}{\partial x} = u \frac{\partial}{\partial u} \quad \text{similarly} \quad \frac{\partial}{\partial y} = v \frac{\partial}{\partial v}$$

we know that

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\partial^2 z}{\partial y \cdot \partial x}$$

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= v \frac{\partial}{\partial v}\left(u \frac{\partial z}{\partial u}\right)$$

$$= uv \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)$$

$$\boxed{\frac{\partial^2 z}{\partial x \cdot \partial y} = uv \frac{\partial^2 z}{\partial u \cdot \partial v}}$$

(5) If $z = f(x,y)$, $x = e^u \cos v$, $y = e^u \sin v$, prove that $x \frac{\partial z}{\partial v} + y \frac{\partial z}{\partial u} = e^{2u} \cdot \frac{\partial z}{\partial y}$.

$\Rightarrow$ solution - given $z = f(x,y)$, $x = e^u \cos v$, $y = e^u \sin v$

we know that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot (e^u \cos v) + \frac{\partial z}{\partial y} (e^u \sin v)$$

$$\text{f} \; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\therefore \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{u}\sin v) + \frac{\partial z}{\partial y}(e^{u}\cos v) \quad ($$

Consider

$$LHS = x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u}$$

$$= (e^{u}\cos v)\left[\frac{\partial z}{\partial x}(-e^{u}\sin v) + \frac{\partial z}{\partial y}(e^{u}\cos v)\right]$$

$$+ (e^{u}\sin v)\left[\frac{\partial z}{\partial x}(e^{u}\cos v) + \frac{\partial z}{\partial y}(e^{u}\sin v)\right]$$

$$= -e^{2u}\cancel{\sin v.\cos v\frac{\partial z}{\partial x}} + e^{2u}\cos^2 v\frac{\partial z}{\partial y}$$

$$+ e^{2u}\cancel{\sin v.\cos v\frac{\partial z}{\partial x}} + e^{2u}\sin^2 v\frac{\partial z}{\partial y}$$

$$= e^{2u}\cos^2 v\frac{\partial z}{\partial y} + e^{2u}\sin^2 v\frac{\partial z}{\partial y}$$

$$= e^{2u}[\cos^2 v + \sin^2 v]\frac{\partial z}{\partial y}$$

$$= e^{2u}\frac{\partial z}{\partial y} \qquad (\cos^2 v + \sin^2 v = 1)$$

$$= RHS$$

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(6) If $z = f(x,y)$ & $u = lx + my$, $v = ly - mx$ then show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = (l^2+m^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + (\right.$

$\Rightarrow$ solution –

we have

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot(l) + \frac{\partial z}{\partial v}(-m) \quad \text{—— (1)}$$

Similarly,

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\therefore \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot(m) + \frac{\partial z}{\partial v}(l) \quad \text{—— (2)}$$

squaring & adding (1) & (2) we get

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left[l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right]^2 + \left[m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}\right]^2$$

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$$= l^2\left(\frac{\partial z}{\partial u}\right)^2 - 2lm\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v} + m^2\left(\frac{\partial z}{\partial v}\right)^2 + m^2\left(\frac{\partial z}{\partial u}\right)^2$$

$$+ 2lm\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v} + l^2\left(\frac{\partial z}{\partial v}\right)^2$$

$$= (l^2+m^2)\left(\frac{\partial z}{\partial u}\right)^2 + (l^2+m^2)\left(\frac{\partial z}{\partial v}\right)^2$$

$$= (l^2+m^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

= RHS

(7) If $x^m + y^m = a^m$, find $\frac{d^2y}{dx^2}$.

⇒ solution – given

$$x^m + y^m = a^m$$

differentiate w.r.t x term by term

$$m.x^{m-1} + my^{m-1}.\frac{dy}{dx} = 0$$

$$\therefore m\left(x^{m-1} + y^{m-1}\frac{dy}{dx}\right) = 0$$

$$\therefore x^{m-1} + y^{m-1}\frac{dy}{dx} = 0 \quad \text{—— (1)}$$

$$\therefore y^{m-1}\frac{dy}{dx} = -x^{m-1}$$

$$\therefore \frac{dy}{dx} = -\frac{x^{m-1}}{y^{m-1}} \quad \text{—— (2)}$$

differentiate again w.r.t 'x'

$$\therefore (m-1).x^{m-2} + (m-1)y^{m-2}.\frac{dy}{dx} + y^{m-1}\left(\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore (m-1)x^{m-2} + (m-1).y^{m-2}\left(-\frac{x^{m-1}}{y^{m-1}}\right) + y^{m-1}.\frac{d^2y}{dx^2} = 0$$

$$\therefore (m-1)x^{m-2}y^{m-1} - (m-1)x^{m-1}y^{m-2} + y^{m-1}y^{m-1}.\frac{d^2y}{dx^2} = 0$$

$$\therefore (m-1)x^{m-2}y^{m-1} - (m-1)x^{m-1}y^{m-2} + y^{2m-2}\frac{d^2y}{dx^2} = 0$$

$$\therefore y^{2m-2}\frac{d^2y}{dx^2} = (m-1)x^{m-1}y^{m-2} - (m-1)x^{m-2}y^{m-1}$$

$$\therefore y^{2m-2}\frac{d^2y}{dx^2} = (m-1)\left[x^{m-1}y^{m-2} - x^{m-2}y^{m-1}\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(m-1)}{y^{2m-2}}\left[x^{m-1}\cdot y^{m-2} - x^{m-2}\cdot y^{m-1}\right]$$ (19)

If $x^x \cdot y^y \cdot z^z = c$, show that $x = y = z$

$$\frac{\partial^2 z}{\partial x . \partial y} = -\left[x \log ex\right]^{-1}$$

solution – Taking logarithm on both sides

$$\therefore x\log x + y\log y + z\log z = \log c$$

diff. partially w.r.t x' treating y constant.

$$\therefore x\cdot\frac{1}{x} + \log x + \left(z\cdot\frac{1}{z} + \log z\right)\frac{\partial z}{\partial x} = 0$$

$$\therefore (1+\log x) + (1+\log z)\frac{\partial z}{\partial x} = 0 \quad \text{—— (1)}$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$$

similarly, $\dfrac{\partial z}{\partial y} = \dfrac{-(1+\log y)}{(1+\log z)}$

diff. (1) partially w.r.t 'y'

$$\therefore (1+\log z)\frac{\partial^2 z}{\partial x . \partial y} + \frac{\partial z}{\partial x}\left[\frac{1}{z}\frac{\partial z}{\partial y}\right] = 0$$

$$\therefore (1+\log z)\frac{\partial^2 z}{\partial x . \partial y} + \frac{(1+\log x)}{(1+\log z)}\cdot\frac{1}{z}\frac{(1+\log y)}{(1+\log z)} = 0$$

put $x = y = z$

$$\therefore (1+\log x)\frac{\partial^2 z}{\partial x . \partial y} + \frac{(1+\cancel{\log x})}{(1+\cancel{\log x})}\cdot\frac{1}{x}\frac{(1+\cancel{\log x})}{(1+\cancel{\log x})} = 0$$

$$\therefore (1+\log x)\frac{\partial^2 z}{\partial x . \partial y} + \frac{1}{x} = 0$$

$$\therefore \frac{\partial^2 z}{\partial x . \partial y} = \frac{-1}{x(1+\log x)}$$

$$\therefore \frac{\partial^2 z}{\partial x . \partial y} = -\frac{1}{x(\log e + \log x)} = -\frac{1}{x\log ex} = -(x\log ex)^{-1}$$

$$= RHS$$

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(9) If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, prove that $\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial x}$

$\Rightarrow$ solution – put $X = xt$, $Y = yt$ we get

$$f(X,Y) = \sin^{-1}\left(\frac{\sqrt{xt}-\sqrt{yt}}{\sqrt{xt}+\sqrt{yt}}\right)$$

$$\therefore f(X,Y) = \sin^{-1}\left[\frac{\sqrt{t}\,(\sqrt{x}-\sqrt{y})}{\sqrt{t}\,(\sqrt{x}+\sqrt{y})}\right]$$

$$\therefore f(X,Y) = \sin^{-1}\left[\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right]$$

$$\therefore f(X,Y) = t^0 \cdot f(x,y)$$

Thus u is homogeneous f^n of $\deg(n)=0$

$\therefore$ By Euler's th^m

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \quad (\because n=0)$$

$$\therefore y\frac{\partial u}{\partial y} = -x\frac{\partial u}{\partial x}$$

$$\therefore \boxed{\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial x}}$$

(10) If $u = x^2 \log\left[\frac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}\right]$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2\log\left[\frac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}\right].$$

$\Rightarrow$ solution – put $X = xt$, $Y = yt$ we get

$$f(X,Y) = x^2t^2\log\left[\frac{\sqrt[3]{yt}-\sqrt[3]{xt}}{\sqrt[3]{yt}+\sqrt[3]{xt}}\right]$$

$$= t^2x^2\log\left[\frac{\sqrt[3]{t}}{\sqrt[3]{t}}\frac{(\sqrt[3]{y}-\sqrt[3]{x})}{(\sqrt[3]{y}+\sqrt[3]{x})}\right]$$

$$\therefore f(X,Y) = t^2 \cdot f(x,y),$$

Thus u is homogeneous f^n of deg. 2

$\therefore$ By Euler's th^m

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cdot x^2\log\left[\frac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}\right]$$

(11) If $u = x^n f(y/x) + y^n g(x/y)$ then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$\Rightarrow$ solution- given $u = x^n f(y/x) + y^n g(x/y)$

put $X = xt$, $Y = yt$

$$\therefore f(X,Y) = x^n t^n f\left(\frac{yt}{xt}\right) + y^n t^n g\left(\frac{xt}{yt}\right)$$

$$= t^n\left[x^n f(y/x) + y^n g\left(\frac{x}{y}\right)\right].$$

$$f(X,Y) = t^n f(x,y)$$

Thus u is homo. f^n of deg. n

$\therefore$ By corollary we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

(12) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

$\Rightarrow$ solution - Here u is not homogeneous f^n of x, y

since $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\Rightarrow \tan u = \frac{x^3+y^3}{x-y}.$$

let $z = \tan u = \frac{x^3+y^3}{x-y} = f(u) = F(x,y)$ say

put $X = xt$, $Y = yt$

$$\therefore F(X,Y) = x^3t^3 + y^3t^3 / xt - yt$$

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$\therefore F(X,Y) = \frac{t^3}{t}\left(\frac{x^3+y^3}{(x-y)}\right)$

$\therefore F(X,Y) = t^2\left[\frac{x^3+y^3}{x-y}\right]$

$F(X,Y) = t^2 F(X,Y)$

Thus, $z = f(u) = \tan u$ is homo. f^n of deg. 2

$\therefore$ By corollary 2.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot\frac{f(u)}{f'(u)} = 2\cdot\frac{\tan u}{\sec^2 u}$$

$$= 2\,\frac{\sin u}{\cos u}\cdot\cos^2 u$$

$$= 2\sin u\cdot\cos u$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u}$$

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13. If $u = \log\left[\frac{x^3+y^3}{x^2+y^2}\right]$ then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -1.$$

$\Rightarrow$ solution - Here u is not homo. f^n of x & y

$u = \log\left[\frac{x^3+y^3}{x^2+y^2}\right]$

$\Rightarrow e^u = \left[\frac{x^3+y^3}{x^2+y^2}\right]$

let $z = e^u = \left[\frac{x^3+y^3}{x^2+y^2}\right] = f(u)$ say

put $X = xt$, $Y = yt$

$$\therefore F(X,Y) = \frac{x^3t^3+y^3t^3}{x^2t^2+y^2t^2} = \frac{t^3}{t^2}\left[\frac{x^3+y^3}{x^2+y^2}\right]$$

$F(X,Y) = t^1\cdot f(X,Y)$

Thus $z = f(u)$ is homo. f^n of deg. 1

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x.\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$ 196

where $g(u) = n\frac{f(u)}{f'(u)} = 1\cdot\frac{e^u}{e^u} = 1$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x.\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=1[0-1]$$

$$= -1$$

14. Show that $JJ' = 1$ for $x = e^v\sec u$, $y = e^v\tan u$.

$\Rightarrow$ solution – we have $\frac{\partial x}{\partial u} = e^v\sec u.\tan u$, $\frac{\partial x}{\partial v} = e^v\sec u$

& $\frac{\partial y}{\partial u} = e^v\sec^2 u$, $\frac{\partial y}{\partial v} = e^v\tan u$

$$\therefore J = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix} e^v\sec u.\tan u & e^v\sec u\\ e^v\sec^2 u & e^v\tan u\end{vmatrix}$$

$$= e^{2v}\sec u.\tan^2 u - e^{2v}\sec^3 u$$

$$= e^{2v}\sec u[\tan^2 u - \sec^2 u]$$

$$= e^{2v}\sec u(-1)$$

$$J = -xe^v$$

Now $\sec^2 u - \tan^2 u = 1$

$$\therefore x^2 e^{-2v} - y^2 e^{-2v} = 1$$

$$\therefore e^{2v} = x^2 - y^2$$

Also $\frac{e^v\sec u}{e^v\tan u} = \frac{x}{y}$

$$\therefore \frac{1/\cos u}{\sin u/\cos u} = \frac{x}{y} \Rightarrow \frac{1}{\sin u} = \frac{x}{y}$$

$$\therefore \boxed{\sin u = y/x}$$

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$\therefore u = \sin^{-1}(y/x)$ & $v = \frac{1}{2}\log(x^2 - y^2)$

$$\therefore \frac{\partial u}{\partial x} = \frac{-y}{x\sqrt{x^2-y^2}}, \quad \frac{\partial u}{\partial y} = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial v}{\partial x} = \frac{x}{x^2-y^2}, \quad \frac{\partial v}{\partial y} = \frac{-y}{x^2-y^2}$$

$$\therefore J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \partial u/\partial y \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \\ \frac{x}{x^2-y^2} & -\frac{y}{x^2-y^2} \end{vmatrix}$$

$$= \frac{y^2}{x(x^2-y^2)^{3/2}} - \frac{x}{(x^2-y^2)^{3/2}}$$

$$= \frac{y^2 - x^2}{x(x^2-y^2)^{3/2}}$$

$$= \frac{-(x^2-y^2)^1}{x(x^2-y^2)^{3/2}}$$

$$= -\frac{1}{x(x^2-y^2)^{1/2}}$$

$$= \frac{-1}{xe^v}$$

$$\therefore JJ' = (xe^v) \times \frac{1}{(-xe^v)} = 1$$

(15) If $x = v^2 + w^2$, $y = w^2 + u^2$, $z = u^2 + v^2$, prove that $JJ' = 1$

$\Rightarrow$ solution – we have $\frac{\partial x}{\partial u} = 0$, $\frac{\partial x}{\partial v} = 2v$, $\frac{\partial x}{\partial w} = 2w$

$$\frac{\partial y}{\partial u} = 2u, \quad \frac{\partial y}{\partial v} = 0, \quad \frac{\partial y}{\partial w} = 2w$$

$$\frac{\partial z}{\partial u} = 2u, \quad \frac{\partial z}{\partial v} = 2v, \quad \frac{\partial z}{\partial w} = 0$$

$$\therefore J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}$$

$= 0(0-4vw) - 2v(0-4uw) + 2w(4uv)$

$= 0 + 8uvw + 8uvw$

$$\boxed{J = 16uvw} \quad \text{--- (1)}$$

Now $y+z-x = w^2+u^2+u^2+v^2-v^2-w^2$

$= 2u^2$

$\therefore \quad 2u^2 = y+z-x$

$\therefore \quad 4u\frac{\partial u}{\partial x} = -1, \quad 4u\frac{\partial u}{\partial y} = 1, \quad 4u\frac{\partial u}{\partial z} = 1$

III'y $2v^2 = x-y+z$ & $2w^2 = x+y-z$

$\therefore 4v\frac{\partial v}{\partial x} = 1$ & $4w\frac{\partial w}{\partial x} = 1$

$4v\frac{\partial v}{\partial y} = -1$ $\quad 4w\frac{\partial w}{\partial y} = 1$

$4v\frac{\partial v}{\partial z} = 1$ $\quad 4w\frac{\partial w}{\partial z} = -1$

$$\therefore J' = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -1/4u & 1/4u & 1/4u \\ 1/4v & -1/4v & 1/4v \\ 1/4w & 1/4w & -1/4w \end{vmatrix}$$

$$= \frac{1}{64uvw}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{1}{64uvw}[4]$$

$$\boxed{J' = \frac{1}{16uvw}} \quad \text{--- (2)}$$

from eq^n (1) & (2), we get

$$JJ' = (16uvw) \times \frac{1}{(16uvw)} = 1$$

(16) If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ then prove that $JJ' = 1$.

$\Rightarrow$ solution – Let $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad \text{— (1)}$$

$x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$

$\therefore \dfrac{\partial x}{\partial u} = 0$ $\quad \therefore \dfrac{\partial y}{\partial u} = \dfrac{\sqrt{w}}{2\sqrt{u}}$ $\quad \therefore \dfrac{\partial z}{\partial u} = \dfrac{\sqrt{v}}{2\sqrt{u}}$

$\dfrac{\partial x}{\partial v} = \dfrac{w}{2\sqrt{vw}} = \dfrac{\sqrt{w}}{2\sqrt{v}}$ $\quad \therefore \dfrac{\partial y}{\partial v} = 0$ $\quad \therefore \dfrac{\partial z}{\partial v} = \dfrac{\sqrt{u}}{2\sqrt{v}}$

$\dfrac{\partial x}{\partial w} = \dfrac{v}{2\sqrt{vw}} = \dfrac{\sqrt{v}}{2\sqrt{w}}$ $\quad \therefore \dfrac{\partial y}{\partial w} = \dfrac{\sqrt{u}}{2\sqrt{w}}$ $\quad \therefore \dfrac{\partial z}{\partial w} = 0$

$\therefore$ eq^n (1) becomes,

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & \sqrt{w}/2\sqrt{v} & \sqrt{v}/2\sqrt{w} \\ \frac{\sqrt{w}}{2\sqrt{u}} & 0 & \sqrt{u}/2\sqrt{w} \\ \sqrt{v}/2\sqrt{u} & \frac{\sqrt{u}}{2\sqrt{v}} & 0 \end{vmatrix}$$

$$= (1/2)(1/2)(1/2) \begin{vmatrix} 0 & \sqrt{w}/\sqrt{v} & \sqrt{v}/\sqrt{w} \\ \frac{\sqrt{w}}{\sqrt{u}} & 0 & \sqrt{u}/\sqrt{w} \\ \frac{\sqrt{v}}{\sqrt{u}} & \frac{\sqrt{u}}{\sqrt{v}} & 0 \end{vmatrix}$$

$$= 1/8 \left[0 - \frac{\sqrt{w}}{\sqrt{v}} \left(0 - \frac{\sqrt{u}}{\sqrt{w}} \cdot \frac{\sqrt{v}}{\sqrt{u}} \right) + \frac{\sqrt{v}}{\sqrt{w}} \left(\frac{\sqrt{w}}{\sqrt{u}} \cdot \frac{\sqrt{u}}{\sqrt{v}} - 0 \right) \right]$$

$$= 1/8 \, [1+1]$$

$$\boxed{J = 1/4} \quad \text{— (1)}$$

let $J' = \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

as

$x = \sqrt{vw}$ $\qquad y = \sqrt{wu}$ $\qquad z = \sqrt{uv}$

$x^2 = vw$ $\qquad y^2 = wu$ $\qquad z^2 = uv$

$\dfrac{y^2z^2}{x^2} = u^2 \quad \therefore \quad u = \sqrt{\dfrac{y^2z^2}{x^2}} = \dfrac{yz}{x}$

$|||^{ly}$

$v = \dfrac{xz}{y}$

$w = \dfrac{xy}{z}$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{-yz}{x^2}, \quad \dfrac{\partial u}{\partial y} = \dfrac{z}{x}, \quad \dfrac{\partial u}{\partial z} = y/x$

$\dfrac{\partial v}{\partial x} = z/y, \quad \dfrac{\partial v}{\partial y} = \dfrac{-xz}{y^2}, \quad \dfrac{\partial v}{\partial z} = x/y$

$\dfrac{\partial w}{\partial x} = y/x, \quad \dfrac{\partial w}{\partial y} = x/y, \quad \dfrac{\partial w}{\partial z} = -xy/z^2$

$\therefore J' = \begin{vmatrix} -yz/x^2 & z/x & y/x \\ z/y & -xz/y^2 & x/y \\ y/z & x/z & \frac{-xy}{z^2} \end{vmatrix}$

$= \dfrac{-yz}{x^2}\left[\dfrac{x^2yz}{y^2z^2} - \dfrac{x^2}{yz}\right] - \dfrac{z}{x}\left[\dfrac{-x}{z} - \dfrac{x}{z}\right] + \dfrac{y}{x}\left[\dfrac{x}{y} - \dfrac{-x}{y}\right]$

$= -1 + 1 + 1 + 1 + 1 + 1$

$= 4 \quad \text{—— (2)}$

from (1) + (2)

$JJ' = 4 \times 1/4 = 1$

(17) If $x+y+z=u$, $y+z=uv$, $z=uvw$, prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

$\Rightarrow$ solution – we have $z=uvw$ & $y+z=uv$

$\therefore$ $y = uv - z = uv - uvw$

Also $x = u-(y+z) = u-uv$

$\therefore$ $\frac{\partial x}{\partial u} = 1-v$, $\frac{\partial x}{\partial v} = -u$, $\frac{\partial x}{\partial w} = 0$

$\frac{\partial y}{\partial u} = v - vw$, $\frac{\partial y}{\partial v} = u-uw$, $\frac{\partial y}{\partial w} = -uv$

$\frac{\partial z}{\partial u} = vw$, $\frac{\partial z}{\partial v} = uw$, $\frac{\partial z}{\partial w} = uv$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)[u^2v - u^2vw + u^2vw] + u[uv^2 - uv^2w + uv^2w]$$

$$= (1-v)[u^2v] + u[uv^2]$$

$$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$$

$$\boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v}$$

18) Find the minimum & maximum values of $x^4+y^4-2x^2+4xy-2y^2$. (2022)

$\Rightarrow$ solution - given f^n $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$

(I) consider $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3-4x+4y = 0 \Rightarrow x^3-x+y=0$ — (1)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3+4x-4y = 0 \Rightarrow y^3+x-y=0$ — (2)

solving for x & y

add (1) & (2) we get

$$\begin{array}{rl} & x^3-x+y = 0 \\ + & y^3+x-y = 0 \\ \hline & x^3+y^3 = 0 \end{array}$$

$\Rightarrow x^3 = -y^3$ $\therefore$ $\boxed{x=-y}$

put value in eq^n (1) we get

$x^3-x+y = 0 \Rightarrow x^3-x-x = 0$

$\Rightarrow x(x^2-2) = 0$

$\Rightarrow x = 0$ or $x^2-2 = 0$

$\Rightarrow x = 0$ or $x = \pm\sqrt{2}$

As $x=-y$

$\therefore y=0$ or $y = \pm\sqrt{2}$.

$\therefore$ stationary pts are $(0,0)$, $(\sqrt{2},-\sqrt{2})$ & $(-\sqrt{2},\sqrt{2})$

(II) find r, s & t

$r = \frac{\partial^2 f}{\partial x^2} = 12x^2-4$, $s = \frac{\partial^2 f}{\partial x \partial y} = 4$, $t = \frac{\partial^2 f}{\partial y^2} = 12y^2-4$

(III) find $rt-s^2$ at stationary pts.

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(i) at $(0,0)$

$$rt - s^2 = (-4)(-4) - (4)^2$$
$$= 0$$

Hence at $(0,0)$ no conclusion.

(ii) at $(\sqrt{2}, -\sqrt{2})$

$$rt - s^2 = (20)(20) - (4)^2$$
$$= 384 > 0 \text{ \& } r = 20 > 0$$

Hence at $(\sqrt{2}, -\sqrt{2})$ given f^n has minimum

$$\text{\& } f_{max} = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2$$
$$= 4 + 4 - 2(2) - 4(2) - 2(2)$$
$$= -8$$

(iii) at $(-\sqrt{2}, \sqrt{2})$.

$$rt - s^2 = 384 > 0 \text{ \& } r = 20 > 0$$

$\therefore$ f^n has minimum value.

$$f = -8.$$

ind the maximum or minimum values of (204)

$x^3+3xy^2-3x^2-3y^2+4$

solution –

we have $f(x,y) = x^3+3xy^2-3x^2-3y^2+4$ — ①

$$\therefore \frac{\partial f}{\partial x} = 3x^2+3y^2-6x$$

$$\frac{\partial f}{\partial y} = 6xy-6y$$

$$\frac{\partial^2 f}{\partial x^2} = 6x-6 \quad , \quad \frac{\partial^2 f}{\partial y^2} = 6x-6$$

$$\frac{\partial^2 f}{\partial x . \partial y} = 6y .$$

solve $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ simultaneously

$\therefore 3x^2+3y^2-6x = 0$ & $6xy-6y = 0$

$\therefore x^2+y^2-2x = 0$ — ② & $xy-y=0$

$y(x-1) = 0$

$\Rightarrow y=0 \;/\; x-1=0$

$\therefore y=0 \;/\; x=1$

when $y=0$, from ② $x^2-2x = 0$

$\therefore x(x-2) = 0$

$\therefore x=0$ <u>or</u> $x=2$

when $x=1$, from ② $1^2+y^2-2(1) = 0$

$\therefore y^2-1 = 0$

$\therefore y^2 = 1$

$\therefore \boxed{y = \pm 1}$

$\therefore$ pts are $(0,0)$, $(2,0)$, $(1,1)$ & $(1,-1)$.

(I) at pt. $(0,0)$ —

(20)

$r = \frac{\partial^2 f}{\partial x^2} = -6, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0, \quad t = \frac{\partial^2 f}{\partial y^2} = -6$

$\therefore rt - s^2 = (-6)(-6) - 0^2 = 36 > 0$

$\therefore f(x,y)$ is stationary at pt. $(0,0)$

But $r = -6 < 0$

$\therefore f(x,y)$ is max. at pt. $(0,0)$

$\therefore f_{max} = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

$= 0 + 0 - 0 - 0 + 4$

$\boxed{f_{max} = 4}$

(II) at pt. $(2,0)$ –

$r = \frac{\partial^2 f}{\partial x^2} = 6x - 6 = 6(2) - 6 = 6$

$s = \frac{\partial^2 f}{\partial x \partial y} = 6y = 6(0) = 0$

$t = \frac{\partial^2 f}{\partial y^2} = 6x - 6 = 6(2) - 6 = 6$

$\therefore rt - s^2 = 6(6) - 0^2 = 36 > 0$

$\therefore f(x,y)$ is stationary at pt. $(2,0)$

Also $r = 6 > 0$

$\therefore f^n$ is minimum at $(2,0)$

& it's minimum value is –

$f_{min} = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

$= (2)^3 + 0 - 3(2)^2 - 0 + 4$

$= 8 - 12 + 4$

$\boxed{f_{min} = 0}$

(III) at pt. $(1,1)$ –

$r = \frac{\partial^2 f}{\partial x^2} = 6x - 6 = 6 - 6 = 0$

$s = \frac{\partial^2 f}{\partial x \partial y} = 6y = 6(1) = 6$

$t = \frac{\partial^2 f}{\partial y^2} = 6x - 6 = 6 - 6 = 0$

19 $\therefore rt - s^2 = 0 - 6^2 = -36 \not> 0$

$\therefore$ we reject this pair.

(IV) at pt $(1, -1)$ -

$r = \frac{\partial^2 f}{\partial x^2} = 6x - 6 = 6(1) - 6 = 0$

$s = \frac{\partial^2 f}{\partial x \cdot \partial y} = 6y = 6(-1) = -6$

$t = \frac{\partial^2 f}{\partial y^2} = 6x - 6 = 6(1) - 6 = 0$

$\therefore rt - s^2 = 0 - 36 = -36 \not> 0$

$\therefore$ we reject this pair.

20 Examine the stationary values -

$f(x, y) = \sin x + \sin y + \sin(x+y)$.

$\Rightarrow$ solution -

given $f(x, y) = \sin x + \sin y + \sin(x+y)$

$\therefore \frac{\partial f}{\partial x} = \cos x + 0 + \cos(x+y) = \cos x + \cos(x+y)$

$\frac{\partial f}{\partial y} = \cos y + \cos(x+y)$

$\frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$

$\frac{\partial^2 f}{\partial x \cdot \partial y} = -\sin(x+y)$

$\frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$.

we now solve $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ simultaneou

$\therefore \cos x + \cos(x+y) = 0$ & $\cos y + \cos(x+y) = 0$

By substraction, we get

$$\cos x + \cos(x+y) = 0$$

$$-\;\underline{\cos y \pm \cos(x+y) = 0}$$

$$\cos x - \cos y = 0$$

$\therefore \cos x = \cos y$

$\Rightarrow \boxed{x = y}$

putting $x = y$ in $\cos x + \cos(x+y) = 0$ we get

$\cos x + \cos(x+x) = 0$

$\therefore \cos x + \cos 2x = 0$

$\therefore \cos x = -\cos(2x)$

$\cos x = \cos(\pi \pm 2x)$

$\therefore \boxed{x = \pi \pm 2x}$

If $x = \pi + 2x$, $x = -\pi \Rightarrow y = x = -\pi$

If $x = \pi - 2x \Rightarrow x = \pi/3 \Rightarrow y = x = \pi/3$

$\therefore$ we get pairs $(-\pi, \pi)$ & $(\pi/3, \pi/3)$.

(I) at pt. $(-\pi, \pi)$ —

$r = \frac{\partial^2 f}{\partial x^2} = -\sin(-\pi) - \sin(-2\pi) = 0$

$s = \frac{\partial^2 f}{\partial x . \partial y} = -\sin(-2\pi) = 0$

$t = \frac{\partial^2 f}{\partial y^2} = -\sin(-\pi) - \sin(-2\pi) = 0$

Hence $rt - s^2 = 0$

$\therefore$ we ~~reject~~ this pair.

(II) at pt. $(\pi/3, \pi/3)$ —

$r = \frac{\partial^2 f}{\partial x^2} = -\sin(\pi/3) - \sin(2\pi/3) = -\sqrt{3}$

$s = \frac{\partial^2 f}{\partial x . \partial y} = -\sin(2\pi/3) = -\sqrt{3}/2$

$t = \frac{\partial^2 f}{\partial y^2} = -\sin(\pi/3) - \sin(2\pi/3) = -\sqrt{3}$

$\therefore rt - s^2 = 3 - 3/4 = 9/4 > 0$

$\therefore f(x, y)$ is stationary at pt. $(\pi/3, \pi/3)$.

Also $r = -\sqrt{3} < 0$

$\therefore$ $f(x,y)$ is maximum at pt. $(\pi/3, \pi/3)$.

& it's max. value is -

$$f_{max} = \sin x + \sin y + \sin(x+y)$$

$$\therefore f_{max} = \sin(\pi/3) + \sin(\pi/3) + \sin(\pi/3 + \pi/3)$$

$$\therefore \boxed{f_{max} = 3\frac{\sqrt{3}}{2}}$$