D.K.T.E. SOCIETY'S

TEXTILE AND ENGINEERING INSTITUTE, ICHALKARANJI.

(*An Autonomous Institute.*)

FIRST YEAR, B.TECH

GET102- ENGINEERING MATHEMATICS-I

TUTORIAL NO. : 04

TOPIC : MATRIX-II

TITLE : LINEAR DEPENDENCE & INDEPENDENCE OF VECTORS, EIGEN VALUES WITH THEIR PROPERTIES.

TUTORIAL NO.04

TOPIC: Matrix -II

TITLE: Linear dependence and independence of vectors, Eigen values with their properties.

Question I : Examine for Linear dependence or indepedence , if dependent find relation between them.

1. $X_1 = [3 \ \ 1 \ \ 1], X_2 = [2 \ \ 0 \ -1] , X_3 = [4 \ \ 2 \ \ 1].$
2. $X_1 = [2 \ -1 \ \ 4], X_2 = [0 \ \ 1 \ \ 2] , X_3 = [6 \ -1 \ \ 16] , X_4 = [4 \ \ 0 \ \ 12].$
3. $X_1 = [2 \ \ 3 \ -1 \ -1], X_2 = [1 \ -1 \ -2 \ -4] , X_3 = [3 \ \ 1 \ \ 3 \ -2] , X_4 = [6 \ \ 3 \ \ 0 \ -7].$
4. $X_1 = [1 \ \ 1 \ \ 3 \ \ 1], X_2 = [2 \ \ 2 \ \ 7 \ -1] , X_3 = [3 \ -1 \ \ 2 \ \ 4].$
5. $X_1 = [2 \ \ 1 \ \ 3 \ \ 2 \ -1], X_2 = [4 \ \ 2 \ \ 1 \ -2 \ \ 3] , X_3 = [0 \ \ 0 \ \ 5 \ \ 6 \ -5] ,$

   $X_4 = [6 \ \ 3 \ -1 \ -6 \ \ 7].$
6. $X_1 = [1 \ \ 2 \ \ 4], X_2 = [2 \ -1 \ \ 3] , X_3 = [0 \ \ 1 \ \ 2] , X_4 = [-3 \ \ 7 \ \ 2].$
7. $X_1 = [1 \ \ 1 \ \ 1], X_2 = [1 \ \ 2 \ \ 3] , X_3 = [2 \ \ 3 \ \ 8] .$
8. $X_1 = [1 \ \ 1 \ \ 1 \ \ 3], X_2 = [1 \ \ 2 \ \ 3 \ \ 4] , X_3 = [2 \ \ 3 \ \ 4 \ \ 7] .$
9. $X_1 = [2 \ -1 \ \ 3 \ \ 2], X_2 = [1 \ \ 3 \ \ 4 \ \ 2] , X_3 = [3 \ -5 \ \ 2 \ \ 2] .$
10. $X_1 = [1 \ \ 2 \ -1 \ \ 0], X_2 = [1 \ \ 3 \ \ 1 \ \ 2] , X_3 = [4 \ \ 2 \ \ 1 \ \ 0] , X_4 = [6 \ \ 1 \ \ 0 \ \ 1].$

Question II : Find eigen values and eigen vector corresponding to the largest eigen value.

11. $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

12. $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

17. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

18. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

19. $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

20. Obtain eigen values of Adj. A and also find eigen vector corresponding to the largest eigen value. Where,

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}.$$

21. Find the sum & the product of the eigen values of the following matrix without solving the characteristic equation, $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Also find eigen values of A^T, A^{-1} & A^3.

TUTORIAL NO-04

Topic Name - Matrix - II

Tut. Name - Linear dependence & independence of vectors, eigen values with their properties

Q Examine for linear dependence or independence if dependent find relⁿ betⁿ them.

① $x_1 = [3 \ 1 \ 1]$, $x_2 = [2 \ 0 \ -1]$, $x_3 = [4 \ 2 \ 1]$

⇒ consider the matrix eqⁿ -

$$k_1 x_1 + k_2 x_2 + k_3 x_3 = 0$$

$$\therefore k_1 [3 \ 1 \ 1] + k_2 [2 \ 0 \ -1] + k_3 [4 \ 2 \ 1] = [0 \ 0 \ 0]$$

$\therefore$

$$3k_1 + 2k_2 + 4k_3 = 0$$
$$k_1 + 0k_2 + 2k_3 = 0$$
$$k_1 - k_2 + k_3 = 0$$

which can be written as -

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by $R_2 - R_1$, $R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by $R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$\therefore k_1 - k_2 + k_3 = 0$, $k_2 + k_3 = 0$ $4k_2 = 0$

$\therefore k_1 - 0 + 0 = 0$ $\therefore \boxed{k_3 = 0}$ $\Rightarrow \boxed{k_2 = 0}$

$\therefore \boxed{k_1 = 0}$

since all k_1, k_2, k_3 are zero.

$\therefore$ The vectors are linearly independent.

(2) $x_1 = [2 \ -1 \ 4]$, $x_2 = [0 \ 1 \ 2]$, $x_3 = [6 \ -1 \ 16]$, $x_4 = [4 \ 0 \ 12]$

$\Rightarrow$ solution - consider the matrix equation.

$$k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 = 0$$

$\therefore k_1 [2 \ -1 \ 4] + k_2 [0 \ 1 \ 2] + k_3 [6 \ -1 \ 16] + k_4 [4 \ 0 \ 12] = 0$

which can be written in matrix form as -

$$\begin{bmatrix} 2 & 0 & 6 & 4 \\ -1 & 1 & -1 & 0 \\ 4 & 2 & 16 & 12 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

by $R_1 \leftrightarrow R_2$ $$\begin{bmatrix} -1 & 1 & -1 & 0 \\ 2 & 0 & 6 & 4 \\ 4 & 2 & 16 & 12 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + 4R_1$

$$\begin{bmatrix} -1 & 1 & -1 & 0 \\ 0 & 2 & 4 & 4 \\ 0 & 6 & 12 & 12 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 \to R_2/2$, $R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} -1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -k_1 + k_2 - k_3 = 0$, $k_2 + 2k_3 + 2k_4 = 0$

$\because r < n$, $n - r = 4 - 2 = 2$ parameters.

put $\boxed{k_4 = t}$, $\boxed{k_3 = s}$

$\therefore$ $k_2 = -2k_3 - 2k_4$

$\boxed{k_2 = -2s - 2t}$

Also $-k_1 + k_2 - k_3 = 0$

$\therefore -k_1 = k_3 - k_2$

$\therefore k_1 = k_2 - k_3$

$= -2s - 2t - s$

$\therefore \boxed{k_1 = -3s - 2t}$

$\therefore$ put these values in (1), we get

$(-3s-2t)x_1 + (-2s-2t)x_2 + sx_3 + tx_4 = 0.$

$\therefore (3s+2t)x_1 + (2s-2t)x_2 - sx_3 - tx_4 = 0$

Hence x_1, x_2, x_3 & x_4 are L.D.

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(3) $x_1 = [2 \ 3 \ -1 \ -1]$, $x_2 = [1 \ -1 \ -2 \ -4]$, $x_3 = [3 \ 1 \ 3 \ -2]$

$x_4 = [6 \ 3 \ 0 \ -7]$

$\Rightarrow$ solution – consider the matrix eq^n –

$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$ —— (1)

$\therefore k_1[2 \ 3 \ -1 \ -1] + k_2[1 \ -1 \ -2 \ -4] + k_3[3 \ 1 \ 3 \ -2] + k_4[6 \ 3 \ 0, -7] = 0$

$\therefore$ $2k_1 + k_2 + 3k_3 + 6k_4 = 0$

$3k_1 - k_2 + 3k_4 + k_3 = 0$

$-k_1 - 2k_2 + 3k_3 + 0k_4 = 0$

$-k_1 - 4k_2 - 2k_3 - 7k_4 = 0$

$$\therefore \begin{array}{cccc} & C_2 & & \\ \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ -1 & 3 & 1 & 3 \\ -2 & -1 & 3 & 0 \\ -4 & -1 & -2 & -7 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$, $R_4 \to R_4 + 4R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 5 & 4 & 9 \\ 0 & 3 & 9 & 12 \\ 0 & 7 & 10 & 17 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2$, $R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 2 & -5 & -3 \\ 0 & 3 & 9 & 12 \\ 0 & 2 & 6 & 8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 2 & -5 & -3 \\ 0 & 3 & 9 & 12 \\ 0 & 0 & 11 & 11 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 \to R_4/11$, $R_3 \to 1/3\, R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 2 & -5 & -3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore k_1 + 2k_2 + 3k_3 + 6k_4 = 0$

$2k_2 - 5k_3 - 3k_4 = 0$

$k_2 + 3k_3 + 4k_4 = 0$

$k_3 + k_4 = 0$

Let $k_4 = t$, $\therefore \boxed{k_3 = -t}$, $k_2 = -t$

& $k_1 = -2k_2 - 3k_3 - 6k_4$

$= -2(-t) - 3(-t) - 6(t)$

$= 2t + 3t - 6t$

$\boxed{k_1 = -t}$

put these values in (1)

$-tx_1 - tx_2 - tx_3 + tx_4 = 0$

$\therefore -x_1 - x_2 - x_3 + x_4 = 0$

$\therefore \boxed{x_4 = x_1 + x_2 + x_3}$

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(4) $x_1 = [1\ 1\ 3\ 1]$, $x_2 = [2\ 2\ 7\ -1]$, $x_3 = [3\ -1\ 2\ 4]$

$\Rightarrow$ solution - consider the matrix eqn -

$k_1x_1 + k_2x_2 + k_3x_3 = 0$

$\therefore k_1[1\ 1\ 3\ 1] + k_2[2\ 2\ 7\ -1] + k_3[3\ -1\ 2\ 4] = [0\ 0\ 0$

$\therefore k_1 + 2k_2 + 3k_3 = 0.$

$k_1 + 2k_2 - k_3 = 0$

$3k_1 + 7k_2 + 2k_3 = 0$

$k_1 - k_2 + 4k_3 = 0$

in matrix form -

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & -1 \\ 3 & 7 & 2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(8)

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -4 \\ 0 & 1 & -7 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$k_1 + 2k_2 + 3k_3 = 0$, $-4k_2 = 0 \Rightarrow \boxed{k_2 = 0}$,

$k_2 - 7k_3 = 0 \Rightarrow \boxed{k_3 = 0}$,

$\therefore k_1 + 2k_2 + 3k_3 \Rightarrow k_1 + 0 + 0 = 0$

$\Rightarrow \boxed{k_1 = 0}$

All k_1, k_2 & k_3 are zero.

$\therefore$ vectors x_1, x_2, x_3 are linearly independent.

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(5) $x_1 = [2\ 1\ 3\ 2\ -1]$, $x_2 = [4\ 2\ 1\ -2\ 3]$,
$x_3 = [0\ 0\ 5\ 6\ -5]$, $x_4 = [6\ 3\ -1\ -6\ 7]$.

$\Rightarrow$ solution - consider the matrix eq^n -

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$$

$\therefore k_1[2\ 1\ 3\ 2\ -1] + k_2[4\ 2\ 1\ -2\ 3] + k_3[0\ 0\ 5\ 6\ -5]$
$+ k_4[6\ 3\ -1\ -6\ 7] = [0\ 0\ 0\ 0]$.

which can be written as -

$$\begin{bmatrix} 2 & 4 & 0 & 6 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 5 & -1 \\ 2 & -2 & 6 & -6 \\ -1 & 3 & -5 & 7 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1/2$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 5 & -1 \\ 2 & -2 & 6 & -6 \\ -1 & 3 & -5 & 7 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - 3R_1$, $R_4 - 2R_1$, $R_5 - R_1$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & -10 \\ 0 & -6 & 6 & -12 \\ 0 & 5 & -5 & 10 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $(-R_3/5)$, $\left(-\frac{R_4}{6}\right)$ & $R_5 + R_3$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_4 - R_3$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $-k_1 + 2k_2 + 3k_4 = 0$, $k_2 - k_3 + 2k_4 = 0$

put $\boxed{k_3 = s}$, $\boxed{k_4 = t}$ $\therefore$ $\boxed{k_2 = s - 2t}$

& $k_1 = -2(s-2t) - 3t = -2s + t$ $\therefore$ $\boxed{k_1 = -2s + t}$

$\therefore$ from (1) we get

$(-2s+t)x_1 + (s-2t)x_2 + sx_3 + tx_4 = 0$

$\therefore$ The vectors are L.D.

(64) (6) $x_1 = [1\ 2\ 4]$, $x_2 = [2\ -1\ 3]$, $x_3 = [0\ 1\ 2]$, $x_4 = [-3\ 7\ 2]$

⇒ solution - consider the matrix eqn.

$$k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 = 0$$

$\therefore k_1[1\ 2\ 4] + k_2[2\ -1\ 3] + k_3[0\ 1\ 2] + k_4[-3\ 7\ 2] = [0\ 0\ 0]$

$\therefore$ $k_1 + 2k_2 + 0k_3 - 3k_4 = 0$

$2k_1 - k_2 + k_3 + 7k_4 = 0$

$4k_1 + 3k_2 + 2k_3 + 2k_4 = 0$

we can write -

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $r = 3$, $n = 4$

$r < n$ $\therefore$ it has infinitely many soln.

put $k_3 = s$, $k_4 = t$.

$k_1 + 2k_2 - 3k_4 = 0$, $-5k_2 + k_3 + 13k_4 = 0$

$\therefore k_1 = 3k_4 - 2k_2$ $\therefore -5k_2 = -s - 13t$

$k_1 = 3t - 2/5[s + 13t]$ $\therefore k_2 = 1/5[s + 13t]$

since all k_i's are non-zero

∴ The vectors are L.D.

∴ eq^n (1) becomes,

$$[3t - 2/5(s+13t)]x_1 + 1/5[s+13t]x_2 + sx_3 + tx_4 = 0.$$

(7) $x_1 = [1\ 1\ 1]$, $x_2 = [1\ 2\ 3]$, $x_3 = [2\ 3\ 8]$

⇒ solution - consider the matrix eq^n -

$$k_1x_1 + k_2x_2 + k_3x_3 = 0$$

$$\therefore k_1[1\ 1\ 1] + k_2[1\ 2\ 3] + k_3[2\ 3\ 8] = [0\ 0\ 0]$$

$$\therefore k_1 + k_2 + 2k_3 = 0$$
$$k_1 + 2k_2 + 3k_3 = 0$$
$$k_1 + 3k_2 + 8k_3 = 0$$

∴ we can write it as -

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∴ $4k_3 = 0$, $k_2 + k_3 = 0$, $k_1 + k_2 + 2k_3 = 0$

⇒ $k_3 = 0$ ⇒ $k_2 = 0$ ⇒ $k_1 = 0$

∵ all k_i's are zero.

∴ The vectors x_1, x_2, x_3 are linearly independent.

(66) (8) $x_1 = [1\ 1\ 1\ 3]$, $x_2 = [1\ 2\ 3\ 4]$, $x_3 = [2\ 3\ 4\ 7]$

→ solution- consider the matrix eqⁿ-

$$k_1x_1 + k_2x_2 + k_3x_3 = 0$$

$\therefore k_1[1\ 1\ 1\ 3] + k_2[1\ 2\ 3\ 4] + k_3[2\ 3\ 4\ 7] = [0\ 0\ 0]$

$\therefore$ $k_1 + k_2 + 2k_3 = 0$

$k_1 + 2k_2 + 3k_3 = 0$

$k_1 + 3k_2 + 4k_3 = 0$

$3k_1 + 4k_2 + 7k_3 = 0$

we can write in matrix form as-

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$, $R_3 - R_1$, $R_4 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_2$, $R_4 - R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore k_1 + k_2 + 2k_3 = 0$, $k_2 + k_3 = 0$

$\therefore$ put $k_3 = s$, $k_2 = -s$

$\therefore k_1 + k_2 + 2k_3 = 0$

$\therefore k_1 - s + 2s = 0$

$\therefore k_1 + s = 0$

$\therefore k_1 = -s$

$\because$ all k_i's are not zero.

$\therefore$ The vectors are linearly dependent.

(9) $x_1 = [2\ -1\ 3\ 2]$, $x_2 = [1\ 3\ 4\ 2]$, $x_3 = [3\ -5\ 2\ 2]$

⇒ solution - consider the matrix eq^n is -

$$k_1 x_1 + k_2 x_2 + k_3 x_3 = 0 \quad \text{--- (1)}$$

$\therefore k_1[2\ -1\ 3\ 2] + k_2[1\ 3\ 4\ 2] + k_3[3\ -5\ 2\ 2] = [0\ 0\ 0$

$$\therefore \quad 2k_1 + k_2 + 3k_3 = 0$$
$$-k_1 + 3k_2 - 5k_3 = 0$$
$$3k_1 + 4k_2 + 2k_3 = 0$$
$$2k_1 + 2k_2 + 2k_3 = 0$$

$\therefore$ we can write it as -

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 2R_1$, $R_3 \rightarrow R_3 + 3R_1$, $R_4 \rightarrow R_4 + 2R_1$

$$\begin{bmatrix} -1 & 3 & -5 \\ 0 & 7 & -7 \\ 0 & 13 & -13 \\ 0 & 8 & -8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2/7$, $R_3/13$, $R_4/8$

$$\begin{bmatrix} -1 & 3 & -5 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$, $R_4 - R_2$

$$\begin{bmatrix} -1 & 3 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -k_1 + 3k_2 - 5k_3 = 0$, $k_2 - k_3 = 0$

let $k_3 = t$ $\therefore \boxed{k_2 = t}$

$\therefore -k_1 + 3t - 5t = 0$

$\therefore -k_1 = 2t$

$\therefore k_1 = -2t$

since all k_i's are not zero.

$\therefore$ The vectors are linearly dependent.

Also put values of k_1, k_2, k_3 in eq^n ①

$(-2t)x_1 + t x_2 + t x_3 = 0$

$\therefore -2x_1 + x_2 + x_3 = 0$

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③ $x_1 = [1\ 2\ -1\ 0]$, $x_2 = [1\ 3\ 1\ 2]$, $x_3 = [4\ 2\ 1\ 0]$, $x_4 = [6\ 1\ 0\ 1]$.

$\Rightarrow$ solution- consider the matrix eq^n-

$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$

$\therefore k_1[1\ 2\ -1\ 0] + k_2[1\ 3\ 1\ 2] + k_3[4\ 2\ 1\ 0] + k_4[6\ 1\ 0\ 1] = 0$

$\therefore k_1 + k_2 + 4k_3 + 6k_4 = 0$

$2k_1 + 3k_2 + 2k_3 + k_4 = 0$

$-k_1 + k_2 + k_3 + 0k_4 = 0$

$0k_1 + 2k_2 + 0k_3 + k_4 = 0$

$\therefore$ we can write,

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2-2R_1,\ R_3+R_1,$

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 2 & 5 & 6 \\ 0 & 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3-2R_2,\ R_4-2R_2$

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & 12 & 23 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$r=n \Rightarrow$ we get unique soln, $R_3/17$

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 1 & 28/17 \\ 0 & 0 & 12 & 23 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 \to R_4 - 12R_3$

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 1 & 28/17 \\ 0 & 0 & 0 & 55/17 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ k_1+k_2+4k_3+6k_4=0$ $\quad k_2-6k_3-11k_4=0$

$k_3+\frac{28}{17}k_4=0$ $\quad$ & $\quad \frac{55}{17}k_4=0$

$\Rightarrow \boxed{k_4=0}$

$\therefore\ k_3+0=0$

$\Rightarrow \boxed{k_3=0}$

$\therefore\ k_2-6(0)-11(0)=0$

$\Rightarrow \boxed{k_2=0}$

Also $\quad k_1+0+0+0=0$

$\Rightarrow \boxed{k_1=0}$

since all k_1, k_2, k_3, k_4 are 0

$\therefore$ vectors are linearly independent.

(70) (10) Find eigen values & eigen vector corresp. to largest eigen values.

(11)
$$\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

$\Rightarrow$ solution - The characteristic eq^n is -

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \quad \text{--- (1)}$$

where $s_1 = a_{11} + a_{22} + a_{33} = 7+6+5 = 18$

s_2 = minor of a_{11} + minor of a_{22} + minor of

$$= \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix}$$

$$= 30 - 4 + 35 + 42 - 4$$

$$= 22 + 77$$

$$s_2 = 99$$

&

$$|A| = \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix}$$

$$= 7(30-4) + 2(-10-0) + 0$$

$$= 182 - 20$$

$$|A| = 162$$

$\therefore$ eq^n (1) becomes,

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

$$\therefore (\lambda - 9)(\lambda^2 - 9\lambda + 18) = 0$$

$$\therefore (\lambda - 9)(\lambda - 6)(\lambda - 3) = 0$$

$$\therefore \lambda = 3, 6, 9.$$

$\because$ largest eigen value = 9.

$\therefore$ eigen vector for $\lambda = 9$

$[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider I^{st} 2 rows,

$$-2x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 - 3x_2 - 2x_3 = 0$$

$\therefore$ by Cramer's rule,

$$\frac{x_1}{\begin{vmatrix} -2 & 0 \\ -3 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -2 & 0 \\ -2 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -3 \end{vmatrix}}$$

$$\therefore \frac{x_1}{+4} = -\frac{x_2}{4} = \frac{x_3}{2}$$

$$\therefore \frac{x_1}{2} = -\frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore x_1 = 2, \quad x_2 = -2, \quad x_3 = 1$$

$\therefore$ eigen vector X for $\lambda = 9$ $= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

---

(12) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

$\Rightarrow$ solution – The char. eq^n is –

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \quad \text{—— (1)}$$

where $s_1 = a_{11} + a_{22} + a_{33} = 8 - 3 + 1 = 6$

s_2 = minor of a_{11} + minor of a_{22} + minor of a_{33}

$$= \begin{vmatrix} -3 & -2 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -8 \\ 4 & -3 \end{vmatrix}$$

$$= -3 - \cancel{8} + \cancel{8} + 6 - 24 + 32$$

$$s_2 = 11$$

$$|A| = \begin{vmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{vmatrix}$$

$$= 8(-3-8) + 8(4+6) - 2(-16+9)$$

$$= -88 + 80 + 14$$

$$|A| = 6$$

$\therefore$ eqn (1) becomes,

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\therefore \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$$

$$\therefore (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\therefore (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\therefore \lambda = 1, 2, 3.$$

$\therefore$ largest eigen value = 3

$\therefore$ for $\lambda = 3$ $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider 1st 2 rows

$$5x_1 - 8x_2 - 2x_3 = 0$$
$$4x_1 - 6x_2 - 2x_3 = 0$$

$\therefore$ by Cramer's rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\therefore \frac{x_1}{16-12} = \frac{-x_2}{-10+8} = \frac{x_3}{-30+32}$$

$$\therefore \frac{x_1}{4} = \frac{-x_2}{-2} = \frac{x_3}{2}$$

$\therefore \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$

$\therefore$ eigen vector $(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$
for $\lambda = 3$

(13) $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$\Rightarrow$ solution - consider the char. eq^n

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \quad \text{--- (1)}$$

where $s_1 = a_{11} + a_{22} + a_{33} = 2+2+2 = 6$

s_2 = minor of a_{11} + minor of a_{22} + minor of a_{33}

$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$

$= 4-1+4-\cancel{1}+4+\cancel{1}$

$= 11$

$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

$= 2(4-1)+1(2+1)+1(-1-2)$

$= 6+3-3$

$|A| = 6$

$\therefore eq^n$ (1) becomes,

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\therefore \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$

$\therefore (\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$

$\therefore (\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\therefore \lambda = 1, 2, 3$

$\because$ largest eigen value $= 3$

$\therefore$ eigen vector for $\lambda = 3$, $[A - \lambda I]X = 0$ gives

(13)

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider Ist & 2nd row

$$-x_1 - x_2 + x_3 = 0$$

$$x_1 - x_2 - x_3 = 0$$

$\therefore$ by cramer's rule

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = -\frac{x_2}{\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix}}$$

$$\therefore \frac{x_1}{2} = \frac{-x_2}{0} = \frac{x_3}{2}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$\therefore x_1 = 1, \; x_2 = 0, \; x_3 = 1$

$\therefore$ eigen vector (X) = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
for $\lambda = 3$

(14) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$\Rightarrow$ solution – The char. eq^n is –

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \quad \text{—— (1)}$$

where $s_1 = a_{11} + a_{22} + a_{33} = 6 + 3 + 3 = 12$

s_2 = minor of a_{11} + minor of a_{22} + minor of a_{33}

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= 9 - 1 + 18 - 4 + 18 - 4$$

$s_2 = 36$

$$|A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(9-1)+2(-6+2)+2(2-6)$$
$$= 48-8-8$$
$$= 32$$

$\therefore$ eq^n (1) becomes,

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\therefore$ $(2-\lambda)(\lambda-2)(\lambda-8) = 0$

$\Rightarrow$ $\lambda = 2, 2, 8.$

$\therefore$ largest eigen value $= 8$

$\therefore$ eigen vector for $\lambda = 8$ -

$[A - \lambda I]\,x = 0$ gives

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider Ist 2 rows

$-2x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 - 5x_2 - x_3 = 0$

by cramer's rule

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\therefore \frac{x_1}{12} = \frac{-x_2}{6} = \frac{x_3}{6}$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore$ $x_1 = 2$, $x_2 = -1$, $x_3 = 1$

$\therefore$ eigen vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

with ... dependence ...

(76) (15) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$\Rightarrow$ solution - The char. eq^n is -

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \text{ ——— (1)}$$

where $s_1 = a_{11} + a_{22} + a_{33} = 2+3+2 = 7$

s_2 = minor of a_{11} + minor of a_{22} + minor of a_{33}

$$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 6-2+4-1+6-2$$

$$= 11$$

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= 2(6-2) - 2(2-1) + 1(2-3)$$

$$= 8-2-1$$

$$= 5$$

$\therefore$ eq^n (1) becomes,

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$\therefore (\lambda-1)(\lambda-1)(\lambda+5) = 0$$

$$\therefore \lambda = 1, 1, 5.$$

$\because$ largest eigen value = 5

$\therefore$ eigen vector for $\lambda = 5$

$[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider I^{st} & 2^{nd} row

$$-3x_1 + 2x_2 + x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

$\therefore$ by Cramer's rule

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore x_1 = 1,\ x_2 = 1,\ x_3 = 1$

$\therefore$ eigen vector $X =$ for $\lambda = 5$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

---

(16) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

$\Rightarrow$ solution - The characteristic eq^n is-

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad \text{--- } (1)$$

where $S_1 = a_{11} + a_{22} + a_{33} = 3 - 3 + 7 = 7$

S_2 = minor of a_{11} + minor of a_{22} + minor of a_{33}

$$= \begin{vmatrix} -3 & -4 \\ 5 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 10 \\ -2 & -3 \end{vmatrix}$$

$$= -21 + 20 + 21 - 15 - 9 + 20$$

$S_2 = 16$

$$|A| = \begin{vmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{vmatrix}$$

$$= 3(-21+20) - 10(-14+12) + 5(-10+9)$$

$\therefore |A| = -3 + 20 - 5$

$\therefore |A| = 12$

$\therefore$ eq^n 1 becomes,

$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

$$\begin{array}{c|cccc} 2 & 1 & -7 & 16 & -12 \\ & - & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\therefore (\lambda-2)(\lambda^2 - 5\lambda + 6) = 0$

$\therefore (\lambda-2)(\lambda-3)(\lambda-2) = 0$

$\therefore \lambda = 2, 2, 3$

$\because$ largest eigen value $(\lambda) = 3$

$\therefore$ The eigen vector for $\lambda = 3$

$[A - \lambda I]X = 0$ gives,

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2/-2$,

$$\begin{bmatrix} 0 & 10 & 5 \\ 1 & 3 & 2 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 10 & 5 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 10 & 5 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider Ist & 2nd row

$$x_1 + 3x_2 + 2x_3 = 0$$
$$0x_1 + 10x_2 + 5x_3 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} 3 & 2 \\ 10 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 3 \\ 0 & 10 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-5} = \frac{-x_2}{5} = \frac{x_3}{10}$$

$$\therefore \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{2}$$

$$\Rightarrow x_1 = -1, \; x_2 = -1, \; x_3 = 2$$

$\therefore$ eigen vector for $\lambda = 3$ $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

---

(17) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$\Rightarrow$ solution - The characteristic eq^n is -

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \quad \text{—— (1)}$$

where $s_1 = a_{11} + a_{22} + a_{33} = 8 + 7 + 3 = 18$

$s_2 =$ minor of a_{11} + minor of a_{22} + minor of a_{33}

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= 21 - 16 + 24 - 4 + 56 - 36$$
$$= 45$$

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 40 - 60 + 20$$

$$= 60 - 60$$

$\therefore |A| = 0$

$\therefore$ eq^n ① becomes,

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\therefore \lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\therefore \lambda(\lambda - 15)(\lambda - 3) = 0$$

$$\therefore \lambda = 0, 3, 15.$$

$\because$ largest eigen value $= 15$

$\therefore$ eigen vector x for $\lambda = 15$ -

$[A - \lambda I]x = 0$ gives

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider Ist & 2nd row

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$\therefore$ by cramer's rule

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\therefore \frac{x_1}{40} = \frac{-x_2}{40} = \frac{x_3}{20}$$

$$\therefore \frac{x_1}{2} = -\frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore x_1 = 2,\ x_2 = -2,\ x_3 = 1$$

$\therefore$ eigen vector for $\lambda = 15$ $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

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(18) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\Rightarrow$ solution – The characteristic eqn is –

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$\therefore (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\therefore \cos^2\theta - (2\lambda\cos\theta) + \lambda^2 + \sin^2\theta = 1$$

$$\therefore \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\therefore \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\lambda = \cos\theta \pm i\sin\theta$$

$\therefore$ largest eigen value $= \cos\theta + i\sin\theta$

$\therefore$ $[A - \lambda I]X = 0$ gives –

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-1/\sin\theta \cdot R_1$, $1/\sin\theta \cdot R_2$ $\quad \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By iR_2

$$\begin{bmatrix} i & 1 \\ i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad ix_1 + x_2 = 0$

i.e. $-x_1 + ix_2 = 0$

$\therefore \quad x_1 - ix_2 = 0$

put $x_2 = t$, $x_1 = it$

put $t = 1$, we get eigen vector $X = \begin{bmatrix} i \\ 1 \end{bmatrix}$

---

(19) $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

$\Rightarrow$ solution – The characteristic eq^n is-

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \quad \text{— (1)}$$

where $s_1 = a_{11} + a_{22} + a_{33} = 4 + 3 - 2 = 5$

$s_2 =$ minor of a_{11} + minor of a_{22} + minor of a_{33}

$= \begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$

$= -6 + 10 - 8 + 6 + 12 - 6$

$= 8$

$|A| = \begin{vmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{vmatrix} = 4(-6+10) - 6(-2+2) + 6(-5+3)$

$= 16 - 12$

$|A| = 4$

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$\therefore$ eqⁿ (1) becomes,

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\therefore (\lambda-1)(\lambda-2)(\lambda-2) = 0$$

$$\therefore \lambda = 1, 2, 2$$

$\because$ largest eigen value $= 2$

$\therefore$ eigen vector for $\lambda = 2$,

$[A - \lambda I] X = 0$ gives

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider Iˢᵗ & 2ⁿᵈ row

$$2x_1 + 6x_2 + 6x_3 = 0$$

$$x_1 + x_2 + 2x_3 = 0$$

$\therefore$ by cramer's rule

$$\frac{x_1}{\begin{vmatrix} 6 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix}}$$

$$\therefore \frac{x_1}{6} = \frac{-x_2}{-2} = \frac{x_3}{-4}$$

$$\therefore \frac{x_1}{3} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$\therefore x_1 = 3, \quad x_2 = 1, \quad x_3 = -2$$

$\therefore$ eigen vector X for $\lambda = 2$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

(84) (20) Obtain eigen values of adj. A & also find eigen vector corresponding to largest eigen value.

where $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$\Rightarrow$ solution - The characteristic eq^n is -

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0 \quad \text{--- (1)}$$

where $s_1 = a_{11} + a_{22} + a_{33} = 3+5+3 = 11$

s_2 = minor of a_{11} + minor of a_{22} + minor of a_{33}

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= 15-1+9-1+15-1$$

$$= 36$$

$$|A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) + 1(-3+1) + 1(1-5)$$

$$= 42 - 2 - 4$$

$$|A| = 36$$

$\therefore$ eq^n (1) becomes,

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\therefore \lambda^3 - 2\lambda^2 - 9\lambda^2 + 18\lambda + 18\lambda - 36 = 0$$

$$\therefore (\lambda-2)(\lambda^2 - 9\lambda + 18) = 0$$

$$\therefore (\lambda-2)(\lambda-3)(\lambda-6) = 0$$

$\therefore$ $\lambda = 2, 3, 6 \Rightarrow$ eigen values of A.

$\because$ largest eigen value = 6

$\therefore$ eigen vector x for $\lambda = 6$

$[A - \lambda I]x = 0$ gives

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider I^{st} & 2^{nd} row

$-3x_1 - x_2 + x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

$\therefore$ by cramer's rule

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\therefore \frac{x_1}{2} = \frac{-x_2}{4} = \frac{x_3}{2}$$

$$\therefore \frac{x_1}{1} = \frac{-x_2}{2} = \frac{x_3}{1}$$

$\therefore x_1 = 1, \; x_2 = -2, \; x_3 = 1$

$\therefore$ eigen vector $(X) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

since eigen values of A are $\lambda_1, \lambda_2, \lambda_3$ then eigen values of adj. A are -

$\lambda_1 \cdot \lambda_2 = 2 \times 3 = 6$, $\lambda_2 \cdot \lambda_3 = 2 \times 6 = 12$, $\lambda_3 \cdot \lambda_1 = 12$.