TUTORIAL NO. 08

TOPIC : Expansion of function

TITLE : Use of Maclaurin's & Taylor's power series.

1. obtain the Maclaurin's expansion of $\tan(\frac{\pi}{4}+x)$.

⇒ We know by Maclaurin's theorem,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad —① $$

Where, $f(x) = \tan(\frac{\pi}{4}+x) \Rightarrow f(0) = \tan(\frac{\pi}{4}) = 1$.

$f'(x) = \cancel{2}\sec^2(\frac{\pi}{4}+x) \Rightarrow f'(0) = \sec^2(\frac{\pi}{4}) = 2$.

$f''(x) = 2\sec^2(\frac{\pi}{4}+x).\tan(\frac{\pi}{4}+x) \Rightarrow f''(0) = 4$.

$f'''(x) = 4\sec^2(\frac{\pi}{4}+x).\tan^2(\frac{\pi}{4}+x) + 2\sec^4(\frac{\pi}{4}+x) \Rightarrow f'''(0) = 16$

substituting above values in eq^n ①, we get

$$\tan(\frac{\pi}{4}+x) = 1 + 2x + \frac{4x^2}{2!} + \frac{16x^3}{3!} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{8x^3}{3} + \cdots$$

2. By maclaurin's series expand $\log(1+e^x)$ in powers of x upto x^4.

⇒ We know by Maclaurin's theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad —①$$

Where,

$f(x) = \log(1+e^x) \Rightarrow f(0) = \log 2$.

$f'(x) = \frac{e^x}{1+e^x} \Rightarrow f'(0) = 1/2$.

$f''(x) = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} \Rightarrow f''(0) = 1/4$.

(17)

$$f'''(x) = \frac{(1+e^x)^2 e^x - 2(1+e^x).e^x e^x}{(1+e^x)^4} \Rightarrow f'''(0) = 0.$$

$$f^{iv}(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x}).3(1+e^x)^2 e^x}{(1+e^x)^6} \Rightarrow f^{iv}(0) = \frac{-1}{8}$$

putting this in equation (1) we get,

$$f(x) = \log 2 + x\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(\frac{1}{4}\right) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}\left(\frac{-1}{8}\right) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$$

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9. show that $e^x(\log(1+x)) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \cdots$

$\Rightarrow$ We know by Maclaurin's theorem,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad (1)$$

Where,

$$f(x) = e^x \log(1+x) \Rightarrow f(0) = 0.$$

$$f'(x) = e^x \log(1+x) + \frac{e^x}{1+x} \Rightarrow f'(0) = 1.$$

$$f''(x) = e^x \log(1+x) + \frac{e^x}{1+x} + \frac{e^x}{1+x} - \frac{e^x}{(1+x)^2} \Rightarrow f''(0) = 1.$$

$$f'''(x) = e^x \log(1+x) + \frac{e^x}{1+x} + \frac{e^x}{1+x} - \frac{e^x}{(1+x)^2} + \frac{e^x}{1+x} - \frac{e^x}{(1+x)^2} + \frac{2e^x}{(1+x)^3}$$

$$\Rightarrow f'''(0) = 2.$$

putting all that values in eq^n (1), we get.

$$f(x) = x + \frac{1}{2!}x^2 + \frac{2x^3}{3!} + \cdots$$

i.e. $e^x \log(1+x) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \cdots$

4. If $x^3 + 2xy^2 - y^3 + x = 1$, Prove that
$y = -1 + x - \frac{x^2}{3} + \cdots$

$\Rightarrow$ We know by Maclaurin's theorem,

$$y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \cdots \quad (1)$$

where,

$x^3 + 2xy^2 - y^3 + x = 1$.

When $x = 0 \Rightarrow -y^3(0) = 1 \Rightarrow y(0) = -1$.

Differentiating w.r.t. x, we get,

$3x^2 + 2y^2 + 4xyy_1 - 3y^2y_1 + 1 = 0$

when $x = 0$. $\Rightarrow y_1(0) = 1$.

Again diff. w.r.t x, we get,

$6x + 4yy_1 + 4x[y_1^2 + yy_2] + 4yy_1 - 6yy_1^2 - 3y^2y_2 = 0$.

when $x = 0 \Rightarrow y_2(0) = -\frac{2}{3}$

substituting all that values in eq^n (1) we get,

$y = -1 + x - \frac{x^2}{3} + \cdots$

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5. Expand in power of x, $e^{x\sin x}$

$\Rightarrow$ Here, $f(x) = e^{x\sin x}$.

Let $y = x\sin x$

$\therefore e^{x\sin x} = e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

$= 1 + x\sin x + \frac{(x\sin x)^2}{2!} + \frac{(x\sin x)^3}{3!} + \cdots$

(18)

8. Prove that, $\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

$\Rightarrow$ $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$.

$$= \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$$

$$= \frac{1}{2}\left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}\right)\right].$$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots\right]$$

$$\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

9. Expand. $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$.

$\Rightarrow$ put $x = \tan u$

$$\therefore \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left[\frac{\sqrt{1+\tan^2 u} - 1}{\tan u}\right]$$

$$= \tan^{-1}\left(\frac{\sec u - 1}{\tan u}\right).$$

$$= \tan^{-1}\left(\frac{1-\cos u}{\sin u}\right).$$

$$= \tan^{-1}\left(\frac{2\sin^2 u/2}{2\sin u/2 \cos u/2}\right)$$

$$= \tan^{-1}\left(\frac{\sin u/2}{\cos u/2}\right).$$

$$= \tan^{-1}(\tan u/2)$$

$$= u/2$$

$$= \frac{1}{2}\tan^{-1}x$$

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right).$$

10. Expand. $\sinh^{-1}(3x+4x^3)$

$\Rightarrow$ put $x = \sinh u$.

$\Rightarrow u = \sinh^{-1}x$.

$$\therefore 3x + 4x^3 = 3\sinh u + 4\sinh^3 u$$

$$= \sinh 3u$$

$$\therefore \sinh^{-1}(3x+4x^3) = \sinh^{-1}(\sinh 3u)$$

$$= 3u$$

$$= 3\sinh^{-1}x$$

$$= 3\left(x - \frac{x^3}{6} + \frac{3x^5}{40} - \cdots\right).$$

11. show that $e^{e^x} = e\left[1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{12}x^4 + \cdots\right]$

$\Rightarrow$ We know by exponential series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= 1 + y, \text{ where, } y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\therefore e^{e^x} = e^{1+y} = e \cdot e^y$$

$$= e\left[1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots\right]$$

$$= e\left[1 + \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \frac{1}{2!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2 + \frac{1}{3!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3 + \cdots\right]$$

$$= e\left[1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) + \frac{x^2}{2}\left(1 + \frac{x}{2} + \frac{x^2}{6}\right)^2 + \frac{x^3}{6}\left(1 + \frac{x}{2} + \frac{x^2}{6} + \cdots\right)^3 + \frac{x^4}{24}\left(1 + \frac{x}{2} + \cdots\right)^4 + \cdots\right]$$

$$= e\left[1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) + \frac{x^2}{2}\left(1 + x + \frac{x^2}{4} + \frac{x^2}{3} + \cdots\right) + \frac{x^3}{6}\left(1 + \frac{3x}{2} + \cdots\right) + \frac{x^4}{24}\left(1 + \frac{x^4}{2} + \cdots\right) + \cdots\right]$$

$$= e\left[1 + x + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{1}{6} + \frac{1}{2} + \frac{1}{6}\right)x^3 + \left(\frac{1}{24} + \frac{7}{24} + \frac{1}{24} + \frac{1}{24}\right)x^4 + \cdots\right]$$

$$= e\left[1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{12}x^4 + \cdots\right]$$

$$\therefore e^{e^x} = e\left[1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{12}x^4 + \cdots\right].$$

Q. Expand $\log\cos(x+\frac{\pi}{4})$, using Taylor's theorem in ascending powers of x.

$\Rightarrow$ We know Taylor's theorem in powers of x.

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \cdots \quad \text{—①}$$

where,

$$f(x+h) = \log\cos(x+\frac{\pi}{4}) \Rightarrow f(x) = \log\cos x \ \& \ h = \frac{\pi}{4}.$$

$$f(x) = \log\cos x \quad \Rightarrow f(\frac{\pi}{4}) = -\log\sqrt{2}$$

$$f'(x) = -\tan x \quad \Rightarrow f'(\frac{\pi}{4}) = -1$$

$$f''(x) = -\sec^2 x \quad \Rightarrow f''(\frac{\pi}{4}) = -2.$$

$$f'''(x) = -2\sec^2 x \tan x \Rightarrow f'''(\frac{\pi}{4}) = -4.$$

substituting these values in eq^n ①, we get,

$$\therefore f(x+h) = \log\cos(x+\frac{\pi}{4})$$

$$= -\log\sqrt{2} + x(-1) + \frac{x^2}{2}(-2) + \frac{x^3}{6}(-4) + \cdots$$

$$= -\log\sqrt{2} - x - x^2 - \frac{2x^3}{3} - \cdots$$

$$\therefore \log\cos(x+\frac{\pi}{4}) = -\log\sqrt{2} - x - x^2 - \frac{2x^3}{3} - \cdots$$

13. Prove that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \cdots$

$\Rightarrow$ Here we have to expand given function $\log(x+h)$ in powers of x.
Hence by Taylor's theorem,

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \cdots \quad \text{--- (1)}$$

Here, $f(x+h) = \log(x+h)$

$\Rightarrow f(x) = \log x \quad \Rightarrow f(h) = \log h$.

As, $f'(x) = \frac{1}{x} \Rightarrow f'(h) = \frac{1}{h}$

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(h) = \frac{-1}{h^2}$

$f'''(x) = \frac{1}{x^3} \Rightarrow f'''(h) = \frac{1}{h^3}$

substituting in eqⁿ (1) we get,

$$\therefore \log(x+h) = \log h + x\frac{1}{h} + \frac{x^2}{2!}\left(\frac{-1}{h^2}\right) + \frac{x^3}{3!}\left(\frac{2}{h^3}\right) + \cdots$$

$$= \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \cdots$$

14. Using Taylor's theorem ~~expansion~~ express $7 + (x+2) + 3(x+2)^3 + (x+2)^4$ in powers of x.

$\Rightarrow$ We know Taylor's theorem in powers of x.

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \cdots \quad \text{--- (1)}$$

Here, $f(x+h) = 7 + (x+2) + 3(x+2)^3 + (x+2)^4$

$\Rightarrow f(x) = 7 + x + 3x^3 + x^4$ & $h = 2$.

$f(x) = 7 + x + 3x^3 + x^4 \Rightarrow \therefore f(2) = 49$

$f'(x) = 1 + 9x^2 + 4x^3 \Rightarrow \therefore f'(2) = 69$

$f''(x) = 18x + 12x^2 \Rightarrow f''(2) = 84$

$f'''(x) = 18 + 24x \Rightarrow f'''(2) = 66$

$f^{iv}(x) = 24 \Rightarrow f^{iv}(2) = 24$

substituting this in eq^n ① we get

$$f(x+h) = 49 + 69x + 84\frac{x^2}{2!} + 66\frac{x^3}{3!} + 24\frac{x^4}{4!}$$

$$= 49 + 69x + 42x^2 + 11x^3 + x^4$$

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15. Expand $\log x$ in powers $(x-2)$ by using Taylor's theorem.

$\Rightarrow$ We know that,

$$f(x) = f(h) + (x-h)f'(h) + \frac{(x-h)^2}{2!}f''(h) + \frac{(x-h)^3}{3!}f'''(h) + \cdots \quad \text{—— ①}$$

put $h = 2$, we get.

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots \quad \text{—— ②}$$

Here, $f(x) = \log x \Rightarrow f(2) = \log 2$

$f'(x) = 1/x \Rightarrow f'(2) = 1/2$

$f''(x) = -1/x^2 \Rightarrow f''(2) = -1/4$

$f'''(x) = 2/x^3 \Rightarrow f'''(2) = 2/8$

substituting this in eq^n ②, we get

$$\log x = \log 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 + \cdots$$

16. Expand $\tan^{-1}x$ in powers of $(x-\pi/4)$.

$\Rightarrow$ We know that Taylor's theorem,

$$f(x) = f(h) + (x-h)f'(h) + \frac{(x-h)^2}{2!}f''(h) + \frac{(x-h)^3}{3!}f'''(h) \quad \text{--- (1)}$$

put $h = \frac{\pi}{4}$

$$f(x) = f(\pi/4) + \left(x-\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right) + \cdots \quad \text{--- (2)}$$

Where, $f(x) = \tan^{-1}x \quad \Rightarrow f(\pi/4) = 1$

$f'(x) = \frac{1}{1+x^2} \quad \Rightarrow f'(\pi/4) = \frac{1}{1+\pi^2/16}$

$f''(x) = \frac{-2x}{(1+x^2)^2} \quad \Rightarrow f''(\pi/4) = \frac{-\pi/4}{[1+(\pi/4)^2]^2}$

substituting in eq. (2) we get

$$\tan^{-1}x = \tan^{-1}\left(\frac{\pi}{4}\right) + \left(x-\frac{\pi}{4}\right)\left(\frac{1}{1+\pi^2/16}\right) + \frac{(x-\pi/4)^2}{2!}\frac{-\pi/2}{[1+(\pi/4)^2]^2} + \cdots$$

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17. Expand $f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8$ in ascending powers of $(x+1)$

$\Rightarrow$ Where,

$f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8$

$\therefore f'(x) = 4(x+2)^3 + 15(x+2)^2 + 12(x+2) + 7$

$f''(x) = 12(x+2)^2 + 30(x+2) + 12$

$f'''(x) = 24(x+2) + 30$

$f^{iv}(x) = 24$

put $x = -1$

$f(-1) = 27, \quad f'(-1) = 38, \quad f''(-1) = 54, \quad f'''(-1) = 54$ (18)

$$\therefore f(x) = f(-1) + (x+1) f'(-1) + \frac{(x+1)^2}{2!} f''(-1) + \frac{(x+1)^3}{3!} f'''(-1) + \cdots$$

$$= 27 + 38(x+1) + 27(x+1)^2 + 9(x+1)^3 + (x+1)^4$$

18. Using Taylor's theorem find $\sqrt{25.15}$

$\Rightarrow$ Let $f(x) = \sqrt{x}$ and $h = 0.15$, $x = 25$

Then, $f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2!} f''(x) + \cdots$ —— (1)

Here, $f(x) = \sqrt{x} \Rightarrow$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f''(x) = -\frac{1}{4} x^{-3/2}$$

$$f'''(x) = \frac{3}{8} x^{-5/2}$$

$$\therefore f(x+h) = \sqrt{x+h} = x^{1/2} + h \frac{1}{2} x^{-1/2} + \frac{h^2}{2!}\left(\frac{-1}{4}\right) x^{-3/2} + \cdots$$

put $x = 25$ & $h = 0.15$, we get,

$$\sqrt{25.15} = 5 + \frac{(0.15)}{2}(5)^{-1} + \frac{(0.15)^2}{2}\left(\frac{-1}{4}\right)5^{-3} + \frac{(0.15)^3}{3!}\frac{3}{8}(5)^{-5}$$

$$= 5 + 0.015 - 0.000225$$

$$= 5.01498$$

19. Apply Taylor's theorem to find approximately the value of $f(11/10)$ where $f(x) = x^3 + 3x^2 + 15x - 10$

$\Rightarrow$ By Taylor's theorem, we have,

$$f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

put $x = 1$, $h = 0.1$

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$f(x) = x^3 + 3x^2 + 15x - 10 \quad \Rightarrow \quad f(1) = 9$

$f'(x) = 3x^2 + 6x + 15 \quad \Rightarrow \quad f'(1) = 24$

$f''(x) = 6x + 6 \quad \Rightarrow \quad f''(1) = 12$

$f'''(x) = 6 \quad \Rightarrow \quad f'''(1) = 6$

$\therefore f\left(\frac{11}{10}\right) = f(1.1)$

$= 9 + \frac{(0.1)}{1} 24 + \frac{(0.1)^2}{2!} 12 + \frac{(0.1)^3}{3!} 6$

$= 11.461$

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20. Expand e^x in powers of $(x-1)$

$\Rightarrow$ Let $f(x) = e^x$ and $a = 1$

$\therefore$ By Taylor's theorem,

$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots \quad \text{—— (1)}$

Here, $f(x) = e^x \quad \Rightarrow \quad f(1) = e$

$f'(x) = e^x \quad \Rightarrow \quad f'(1) = e$

$f''(x) = e^x \quad \Rightarrow \quad f''(1) = e$

$f'''(x) = e^x \quad \Rightarrow \quad f'''(1) = e$

using all that values in eq^n (1) we get.

$e^x = e + (x-1) e + \frac{(x-1)^2}{2!} e + \frac{(x-1)^3}{3!} e + \cdots$

$= e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right]$