TUTORIAL NO. 03

TOPIC : Matrix - I

TITLE : Use of matrix equations $AX = 0$ and $AX = B$ to solve homogeneous and non-homogeneous linear equations based on rank and number of variables

1. Examine for consistency and solve, if consistent

$$x_1 + x_2 + x_3 = -3,$$
$$3x_1 + x_2 - 2x_3 = -2,$$
$$2x_1 + 4x_2 + 7x_3 = 7$$

$\Rightarrow$ the given equations can be written in matrix form as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$$

Augmented matrix $[A:B]$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

By $R_2 - 3R_1$, $R_3 - 2R_1$

$$[A:B] \simeq \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right]$$

By $R_3 + R_2$

$$[A:B] \simeq \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

which is in echelon form and $\rho(A) = 2$ and $\rho[A:B] = 3$

$\therefore \rho(A) \neq \rho[A:B]$.

$\therefore$ the system is inconsistent and has no solution.

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2. Examine for consistency & solve, if consistent,

$$x_1 + x_2 + x_3 = 3,$$
$$2x_1 - x_2 + 3x_3 = 1,$$
$$4x_1 + x_2 + 5x_3 = 2,$$
$$3x_1 - 2x_2 + x_3 = 4.$$

$\Rightarrow$ the given equations can be written in matrix form as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & 5 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

(10) The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 1 \\ 4 & 1 & 5 & 2 \\ 3 & 2 & 1 & 4 \end{array}\right]$$

By $R_2 - 2R_1,\ R_3 - 4R_1,\ R_4 - 3R_1$

$$[A:B] \approx \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & -3 & 1 & -10 \\ 0 & -5 & -2 & -5 \end{array}\right]$$

By $R_3 - R_2,\ -\frac{1}{5}R_4$

$$[A:B] \approx \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & -5 \\ 0 & 1 & 2/5 & 1 \end{array}\right]$$

By $3R_4 + R_2$

$$[A:B] \approx \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 11/5 & -2 \end{array}\right]$$

By R_{34}

$$[A:B] \approx \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 11/5 & -2 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

which is echelon form and $\rho(A) = 3 \neq \rho[A:B] = 4$.

$\therefore \rho(A) \neq \rho[A:B]$.

$\therefore$ The system is inconsistent and has no solution.

3. Test for consistency & if consistent solve

$x_1 + 2x_2 + 2x_3 = 1,$

$2x_1 + 2x_2 + 3x_3 = 3,$

$x_1 - x_2 + 3x_3 = 5$

$\Rightarrow$ The given equations can be written in matrix form as,

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Augmented matrix is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 2 & 2 & 3 & 3 \\ 1 & -1 & 3 & 5 \end{array}\right]$$

By $R_2 - 2R_1,\ R_3 - R_1$

$$[A:B] \approx \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & -3 & 1 & 4 \end{array}\right]$$

By $R_2 - R_3$

$$[A:B] \approx \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & 1 & -2 & : & -3 \\ 0 & -3 & 1 & : & 4 \end{bmatrix}$$

By $R_3 + 3R_2$

$$[A:B] \approx \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & -5 & : & -5 \end{bmatrix}$$

By $\frac{-1}{5} R_3$

$$[A:B] \approx \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

Which is echelon form, and $\rho(A) = \rho[A:B] = 3$

Hence system is consistent and also $\rho(A) = \rho[A:B] = n = 3$

$\therefore$ The system has unique solⁿ.

$$\therefore \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 + 2x_3 = 1$

$x_2 - 2x_3 = 1$.

$x_3 = 1$

$\Rightarrow x_1 = 1, \quad x_2 = -1, \quad x_3 = 1$

2. Show that the equations (4)

$x_1 + 2x_2 + 3x_3 = 6$,

$2x_1 + 3x_2 = 11$,

$4x_1 + x_2 - 5x_3 = -3$.

are consistent & hence solve these equations.

$\Rightarrow$ The given equations can be written in matrix form as,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 4 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -3 \end{bmatrix}.$$

Augmented matrix is,

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 2 & 3 & 0 & : & 11 \\ 4 & 1 & -5 & : & -3 \end{bmatrix}$$

By $R_2 - 2R_1$, $R_3 - 4R_1$.

$$[A:B] \approx \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -6 & : & -1 \\ 0 & -7 & -17 & : & -27 \end{bmatrix}$$

By $R_3 - 7R_2$

$$[A:B] \approx \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -6 & : & -1 \\ 0 & 0 & 25 & : & -20 \end{bmatrix}$$

this is in echelon form &

$\rho(A) = \rho[A:B] = 3$.

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Hence system is consistent and also $\rho(A) = \rho[A:B] = n = 3$.

$\therefore$ The system has unique sol^n.

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -20 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 + 3x_3 = 6$$

$$x_2 + 6x_3 = 1$$

$$25x_3 = -20.$$

$$\therefore x_1 = \frac{-16}{5}, \quad x_2 = \frac{29}{5}, \quad x_3 = \frac{-4}{5}.$$

5. Is the following system of equations consistent? If so solve it

$$x_1 + x_2 + x_3 = 1$$

$$x_1 + 2x_2 + 4x_3 = 2$$

$$x_1 + 4x_2 + 10x_3 = 4.$$

$\Rightarrow$ The given equations can be written in matrix form as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

The augmented matrix is,

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 4 & 10 & 4 \end{array}\right]$$

By $R_2 - R_1$, $R_3 - R_1$

$$[A:B] \approx \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 9 & 3 \end{array}\right]$$

By $R_3 - 3R_2$

$$[A:B] \approx \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which is in echelon form & $\rho(A) = \rho[A:B] = 2$

Hence system is consistent and also $\rho(A) = \rho[A:B] = 2$ and $n = 3$.

$\therefore r < n$

Thus the given system has infinite solutions and $n - r = 3 - 2 = 1$, the number of unknown para.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 + x_3 = 1$, $x_2 + 3x_3 = 1$.

By putting $x_3 = t$, we get

$x_1 = 2t$, $x_2 = 1 - 3t$, $x_3 = t$.

6. Examine for consistency and solve, if consistent.

$x_1 - x_2 + 2x_3 = 4$

$3x_1 + x_2 + 4x_3 = 6$

$x_1 + x_2 + x_3 = 1$.

$\Rightarrow$ The given equations can be written in matrix form as

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$$

the augmented matrix is,

$$[A:B] = \left[\begin{array}{ccc:c} 1 & -1 & 2 & 4 \\ 3 & 1 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

By $R_2 - 3R_1$, $R_3 - R_1$

$$[A:B] \approx \left[\begin{array}{ccc:c} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 2 & -1 & -3 \end{array}\right]$$

By $2R_3 - R_2$

$$[A:B] \approx \left[\begin{array}{ccc:c} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which is in echelon form & $\rho(A) = \rho[A:B] = 2$. Hence system is consistent and also $\rho(A) = \rho[A:B] = 2$ & $n = 3$

$\therefore r < n$

Thus the given system has infinite solutions and $n - r = 3 - 2 = 1$ the number of unknown parameters.

$$\therefore \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 - x_2 + 2x_3 = 4$

$4x_2 - 2x_3 = -6$.

By putting $x_3 = t$ we get

$x_1 = \dfrac{3-3t}{2}$, $x_2 = \dfrac{-3+t}{2}$, $x_3 = t$

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7. Examine for consistency & solve, if consistent.

$2x_1 - x_2 - x_3 = 2$,

$x_1 + 2x_2 + x_3 = 2$.

$4x_1 - 7x_2 - 5x_3 = 2$.

$\Rightarrow$ The given equations can be written in matrix form as

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Augmented matrix is,

$$[A:B] \approx \left[\begin{array}{ccc:c} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right]$$

By R_{12}

$$[A:B] \simeq \begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 1 & -1 & -1 & \vdots & 2 \\ 4 & -7 & -5 & \vdots & 2 \end{bmatrix}$$

By $R_2 - 2R_1$, $R_3 - 4R_1$

$$[A:B] \simeq \begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 0 & -5 & -3 & \vdots & -2 \\ 0 & -15 & -9 & \vdots & -6 \end{bmatrix}$$

By $R_3 - 3R_2$

$$[A:B] \simeq \begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 0 & -5 & -3 & \vdots & -2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

which is in echelon form and $\rho(A) = \rho[A:B] = 2$. Hence system is consistent and also $\rho(A) = \rho[A:B] = 2 =$ and $n = 3$.

$\therefore r < n$.

thus given system has infinite solutions and $n - r = 3 - 2 = 1$ the number of unknown parameters

$$\therefore \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 + 2x_2 + x_3 = 2$

$-5x_2 - 3x_3 = -2$.

put $x_3 = t$ we get,

$x_1 = (6+t)/5$ $x_2 = (2-3t)/5$.

8. Is the following system of equations consistent? If so solve it.

$x_1 + x_2 + x_3 = 6$, $x_1 - x_2 + 2x_3 = 5$

$3x_1 + x_2 + x_3 = 8$, $2x_1 - 2x_2 + 3x_3 = 7$

$\Rightarrow$ The given equation can be written in matrix form as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix}$$

The augmented matrix is,

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & -1 & 2 & \vdots & 5 \\ 3 & 1 & 1 & \vdots & 8 \\ 2 & -2 & 3 & \vdots & 7 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - 3R_1$, $R_4 - 2R_1$

$$[A:B] \simeq \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & -2 & 1 & \vdots & -1 \\ 0 & -2 & -2 & \vdots & -10 \\ 0 & -4 & 1 & \vdots & -5 \end{bmatrix}$$

By $R_3 - R_2$, $R_4 - 2R_2$

$$[A:B] \simeq \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & -2 & 1 & \vdots & -1 \\ 0 & 0 & -3 & \vdots & -9 \\ 0 & 0 & -1 & \vdots & -3 \end{bmatrix}$$

By $(2R_4 - R_3)$

$$[A:B] \approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in echelon form & $\rho(A) = \rho[A:B] = 3$. Hence the system is consistent and also $\rho(A) = \rho[A:B] = n = 3$. the system has unique solution.

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -9 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 + x_3 = 6, \quad -2x_2 + x_3 = -1$

$-3x_3 = -9.$

$\Rightarrow x_1 = 1, \ x_2 = 2 \ \& \ x_3 = 3.$

9. Find the value of λ for which the following system of equations are consistent.

$3x_1 - 2x_2 + 2x_3 = 3,$

$x_1 + \lambda x_2 - 3x_3 = 0$

$4x_1 + x_2 + 2x_3 = 7.$

$\Rightarrow$ the given equation can be written in matrix form as,

$$\begin{bmatrix} 3 & -2 & 2 \\ 1 & \lambda & -3 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$$

Augmented matrix is

$$[A:B] \approx \left[\begin{array}{ccc|c} 3 & -2 & 2 & 3 \\ 1 & \lambda & -3 & 0 \\ 4 & 1 & 2 & 7 \end{array}\right]$$

By R_{12}

$$[A:B] \approx \left[\begin{array}{ccc|c} 1 & \lambda & -3 & 0 \\ 3 & -2 & 2 & 3 \\ 4 & 1 & 2 & 7 \end{array}\right]$$

By $R_2 - 3R_1, \ R_3 - 4R_1$

$$[A:B] \left[\begin{array}{ccc|c} 1 & \lambda & -3 & 0 \\ 0 & -2-3\lambda & 11 & 3 \\ 0 & 1-4\lambda & 14 & 7 \end{array}\right]$$

As given system of equation is consistent, therefore matrix must be in echelon form & this is possible only when $1 - 4\lambda = 0 \Rightarrow \lambda = 1/4$.

10. For what values of λ and μ the equations

$x_1 + x_2 + x_3 = 6$

$x_1 + 2x_2 + 3x_3 = 10$

$x_1 + 2x_2 + \lambda x_3 = \mu$

have i) Unique solution

ii) No solution

iii) infinite number of solutions

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$\Rightarrow$ The given equations can be written in matrix form as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

the augmented matrix is,

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

By $R_2 - R_1$, $R_3 - R_1$.

$$[A:B] \simeq \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

By $R_3 - R_2$

$$[A:B] \simeq \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

i) For a given system of equation have a unique solution. $\rho(A) = \rho[A:B] = n = 3$

$\therefore \lambda - 3 \neq 0$ and μ may have any value.

ii) For a given system of equation has no solution if $\rho(A) \neq \rho[A:B]$

$\therefore \lambda - 3 = 0$ & $\mu - 10 \neq 0$

i.e. $\lambda = 3$ and $\mu \neq 10$.

iii) For a given system of equation has infinite number of solutions if

$\rho(A) = \rho[A:B] = r < n$

i.e. $\lambda - 3 = 0$, $\mu - 10 = 0$

i.e. $\lambda = 3$, $\mu = 10$.

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11. Test whether the following system of equations passesses a non trivial solution.

$$x_1 + x_2 + 2x_3 + 3x_4 = 0$$
$$3x_1 + 4x_2 + 7x_3 + 10x_4 = 0$$
$$5x_1 + 7x_2 + 11x_3 + 17x_4 = 0$$
$$6x_1 + 8x_2 + 13x_3 + 16x_4 = 0$$

$\Rightarrow$ The given equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 3R_1$, $R_3 - 5R_1$, $R_4 - 6R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 2R_2$, $R_4 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_4 - R_3$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form and $\rho(A) = 4$, $n = 4$.

$\therefore \rho(A) = r = n$.

Hence the given homogeneous system of equations has trivial solution.

$\therefore x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0.$

12. Solve the following system of equations,

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_1 + 3x_2 + x_3 = 0$$
$$4x_1 + 5x_2 + 4x_3 = 0$$
$$x_1 + x_2 - 2x_3 = 0$$

$\Rightarrow$ the given equations can be written in matrix form as

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 2R_1$, $R_3 - 4R_1$, $R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -3 & -8 \\ 0 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 3R_2$, $R_4 - R_2$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 7 \\ \hdashline 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form, and $\rho(A) = 3$, $n = 3$.

$\therefore \rho(A) = r = n$.

Hence the given homo. system of equations has trivial solution.

$\therefore x_1 = 0,\ x_2 = 0,\ x_3 = 0.$

13. Solve the following system of equations

$$4x_1 - x_2 + x_3 = 0$$
$$x_1 + 2x_2 - x_3 = 0$$
$$3x_1 + x_2 + 5x_3 = 0.$$

$\Rightarrow$ The given equations can be written in matrix form as

$$\begin{bmatrix} 4 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By R_{12},

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 4R_1$, $R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -9 & 5 \\ 0 & -5 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\frac{-1}{9} R_2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5/9 \\ 0 & -5 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 + 5R_2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5/9 \\ 0 & 0 & 47/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is in echelon form and $\rho(A) = 3$, $n = 3$.

$\therefore \rho(A) = r = n$.

Hence the given homo. system of equations has trivial solution

$\therefore x_1 = 0, \; x_2 = 0, \; x_3 = 0.$

14. Solve the homogeneous system of equations,

$$4x_1 + 3x_2 - x_3 = 0$$
$$3x_1 + 4x_2 + x_3 = 0$$
$$x_1 - x_2 - 2x_3 = 0$$
$$5x_1 + x_2 - 4x_3 = 0.$$

$\Rightarrow$

The given equations can can be written in matrix form as,

$$\begin{bmatrix} 4 & 3 & -1 \\ 3 & 4 & 1 \\ 1 & -1 & -2 \\ 5 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By R_{13}

$$\begin{bmatrix} 1 & -1 & -2 \\ 3 & 4 & 1 \\ 4 & 3 & -1 \\ 5 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 3R_1$, $R_3 - 4R_1$, $R_4 - 5R_1$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 7 & 7 \\ 0 & 7 & 7 \\ 0 & 6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\frac{1}{7}R_2$, $\frac{1}{7}R_3$, $\frac{1}{6}R_4$.

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - R_2$, $R_4 - R_2$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is echelon form and $\rho(A) = 2$ & $n = 3$ $\therefore r < n$.

$\therefore$ The system have infinite number of solutions.

$\therefore x_1 = t$, $x_2 = -t$, & $x_3 = t$.

15. Solve the system of eq^n

$$x_1 + 2x_2 + 3x_3 + x_4 = 0$$
$$x_1 + x_2 - x_3 - x_4 = 0$$
$$3x_1 - x_2 + 2x_3 + 3x_4 = 0.$$

$\Rightarrow$ the given equations can be written in matrix form a

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 3 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & -2 \\ 0 & -7 & -7 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 7R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & 21 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which is in echelon form,

$\rho(A) = 3$ and $n = 4$

$\therefore \rho(A) = r < n$.

$\therefore$ The system have an infinite number of solutions.

$\therefore x_4 = t$, $x_3 = \frac{-14t}{21}$, $x_2 = \frac{2}{3}t$

& $x_1 = -\frac{1}{[illegible]}t$

16. solve by matrix method

$x+y+2z=0,$
$x+2y+3z=0,$
$x+3y+4z=0,$
$3x+4y+7z=0.$

$\Rightarrow$ the given system of eqⁿ can be written in matrix form as,

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - R_1$, $R_4 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 2R_2$, $R_4 - R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the matrix is $2 < 3$, the no. of variables.
$\therefore$ The system has non-trivial solution. The number of independent solⁿ is $3-2=1$.

From the above eqⁿ we get
$x+y+2z=0, \quad y+z=0$
putting $z=-t$, we get
$y=t$ & $x=t$
$\therefore x=t,\ y=t$ & $z=-t$

17. solve the following system of equations,

$3x+y-5z=0,$
$5x+3y-6z=0.$
$x+y-2z=0$
$x-5y+z=0.$

$\Rightarrow$ Taking the third & fourth equations as first & second then we have,

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 1 \\ 3 & 1 & -5 \\ 5 & 3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - 3R_1$, $R_4 - 5R_1$.

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -6 & 3 \\ 0 & -2 & 1 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\frac{1}{3}R_3$, $\frac{1}{2}R_4$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By R_{24}, R_{34}

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank $= 3$ & no. of variables $= 3$. Hence only possible solution is the trivial solution.

$\therefore x = 0,\ y = 0$ & $z = 0$

18. For what value of λ the equations
$x + y + z = 1,$
$x + 2y + 4z = \lambda,$
$x + 4y + 10z = \lambda^2$
have a solution and solve them completely in each case.

$\Rightarrow$ the given system of eq^n can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 1 \end{bmatrix}$$

By $R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 3\lambda + 2 \end{bmatrix}$$

The given equations will be consistent if the rank of A = rank of $[A:B]$.

This requires $\lambda^2 - 3\lambda + 2 = 0$

i.e. $(\lambda - 2)(\lambda - 1) = 0$

$\therefore \lambda = 2$ or $\lambda = 1$.

(2) a) If $\lambda = 2$ the eqⁿ ① becomes,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore x + y + z = 1, \quad y + 3z = 1.$

putting $z = t$, we get,

$x = 2t, \quad y = 1 - 3t, \quad z = t.$

b) If $\lambda = 1$ the eqⁿ ① becomes,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x + y + z = 1, \quad y + 3z = 0.$

putting $z = t$, we get

$x = 1 + 2t, \quad y = -3t, \quad z = t.$

Q9. For what value of λ the equations

$3x - 2y + \lambda z = 1,$

$2x + y + z = 2,$

$x + 2y - \lambda z = -1$

will have no unique solution? will the equations have any solution for this value of λ?

$\Rightarrow$ We have,

Taking the equations in reverse order.

$$\begin{bmatrix} 1 & 2 & -\lambda \\ 2 & 1 & 1 \\ 3 & -2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

By $R_2 - 2R_1$, $R_3 - 3R_1$,

$$\begin{bmatrix} 1 & 2 & -\lambda \\ 0 & -3 & 1+2\lambda \\ 0 & -8 & 4\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix} \quad \text{①}$$

$\therefore$ The equations have unique solutions if the coefficient matrix is non-singular.

$\therefore -12\lambda + 8 + 16\lambda \neq 0.$

$4\lambda \neq -8 \Rightarrow \lambda \neq -2$

$\therefore$ The equations have unique solutions if $\lambda \neq 2$ and the equations have no unique solution if $\lambda = -2$.

Further if $\lambda = -2$ we have from ①.

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -8 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}$$

By $R_3 - \frac{8}{3} R_3$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -20/3 \end{bmatrix}$$

$\therefore 0x + 0y + 0z = -20/3$ which is absured.

$\therefore$ The equations are inconsistent

For $\lambda = -2$ there is no solution.